

DISTINGUISHED THETA REPRESENTATIONS FOR BRYLINSKI-DELIGNE COVERING GROUPS

FAN GAO

To Professor Freydoon Shahidi on his 70th birthday

ABSTRACT. For Brylinski-Deligne covering groups of an arbitrary split reductive group, we consider theta representations attached to certain exceptional genuine characters. The goal of the paper is to investigate the dimension of the space of Whittaker functionals of a theta representation. In particular, we determine when the dimension is exactly one, in which case the theta representation is called distinguished. For this purpose, we first give effective lower and upper bounds for the dimension of Whittaker functionals for general theta representations. As a consequence, the dimension in many cases can be reduced to simple combinatorial computations, e.g., the Kazhdan-Patterson covering groups of the general linear groups, or covering groups whose complex dual groups (à la Finkelberg-Lysenko-McNamara-Reich) are of adjoint type. In the second part of the paper, we consider covering groups of certain simply-connected groups and give necessary and sufficient condition for the theta representation to be distinguished. There are subtleties arising from the relation between the rank and the degree of the covering group. However, in each case we will determine the exceptional character such that its associated theta representation is distinguished.

1. INTRODUCTION AND MAIN RESULTS

1.1. Introduction. Let F be a non-archimedean local field of characteristic 0 and residue characteristic p . For a natural number $n \geq 1$, we assume that F^\times contains the full subgroup of the n -th roots of unity, which is then denoted by μ_n . Let \mathbb{G} be a connected split reductive group over F , and let $G := \mathbb{G}(F)$ be its rational points. One of the central ingredients in the study of irreducible admissible representation of G is the uniqueness of Whittaker functionals (cf. [Rod]). For instance, this uniqueness is crucial in the Langlands-Shahidi theory of L -functions ([Sha]) for the so-called generic representations of G , i.e., those with nontrivial Whittaker functionals, unique up to a scalar factor.

In this paper, we work with the Brylinski-Deligne n -fold covering groups $\overline{G}^{(n)}$ of G , see §2.1 for details on such covering groups. We may write $\overline{G}^{(n)}$ and \overline{G} interchangeably if no confusion arises. For simplicity, the phrase *covering groups* in this paper is used to refer to the Brylinski-Deligne covering groups. For this purpose, it is noteworthy to mention that the Brylinski-Deligne framework is quite encompassing and contains almost all classically interesting covering groups ([S], [Mo] and [Ma]), in particular the Matsumoto covering groups of semisimple simply-connected groups in [Mo] and the Kazhdan-Patterson covering groups $\overline{\mathrm{GL}}_r^{(n)}$ of GL_r in [KP].

For covering groups, the uniqueness of Whittaker functionals for genuine representations of $\overline{G}^{(n)}$ holds rarely and one nontrivial example is the classical double cover $\overline{\mathrm{Sp}}_{2r}^{(2)}$ of the symplectic group Sp_{2r} , see [Szp2]. This uniqueness plays pivotal role in the work of Szpruch (cf. [Szp1]-[Szp3]) generalizing the method of Langlands-Shahidi to $\overline{\mathrm{Sp}}_{2r}^{(2)}$. Besides this special family of examples (and that in [GHPS]), the uniqueness of Whittaker functionals fails widely, and one almost never expects such uniform property for all genuine representations of a general covering group. For example, it is well-known that certain theta representations for the Kazhdan-Patterson coverings $\overline{\mathrm{GL}}_r^{(n)}$ of GL_r could have high dimensional space of Whittaker

2010 *Mathematics Subject Classification.* Primary 11F70; Secondary 22E50.

Key words and phrases. Brylinski-Deligne covering groups, Theta representations, Whittaker functionals, distinguished characters, dual groups.

functionals (cf. [KP]). In fact, such theta representations show that the analogous standard module conjecture (which is a theorem for linear algebraic groups by [CS]) does not hold for covering groups.

The failure of the uniqueness of Whittaker functionals for general genuine representations of covering groups, however, has been the source of both obstacles and inspirations to some advancement of the representation theory of such groups. On one hand, for instance, it is not a priori clear how to generalize the Langlands-Shahidi theory of L -functions to covering groups because of the non-uniqueness of Whittaker functionals for unramified principal series representations. Equivalently, it is essentially due to the fact that the analogous Casselman-Shalika formula for covering groups as in [CO] and [Mc2] is vector-valued, whereas for linear algebraic groups it is scalar-valued (cf. [CS1]).

On the other hand, there are various streams of rich theories stemming from the non-existence or multi-dimensionality of Whittaker functionals. For instance, for genuine representations of covering groups without Whittaker functionals, one may consider semi-Whittaker functionals as in [Tak] or degenerate Whittaker-functionals [MW], which interact fruitfully with the arithmetic and character theory of the representations. Meanwhile, the theory of unipotent orbit as discussed in [Gin] and [FG1]-[FG3] for instance also rectify the situation in the absence of Whittaker functionals. In the latter case where multi-dimensionality holds, the theory of multiple Weyl Dirichlet series makes deep and fascinating connections between representation theory of covering groups, quantum physics and statistical mechanics etc, see [BBF], [BFH] and [BFG] for some of the ideas involved. In particular, the book [BFG] contains several excellent expository articles on multiple Dirichlet series.

Nevertheless, in this paper we consider only the so-called theta representations $\Theta(\overline{G}^{(n)}, \overline{\chi})$ which appear as the local representations for the residue of the Borel Eisenstein series (see Definition 2.1). Moreover, we are mostly interested in determining when the space of Whittaker functionals for $\Theta(\overline{G}^{(n)}, \overline{\chi})$ has dimension one, in which case $\Theta(\overline{G}^{(n)}, \overline{\chi})$ is called distinguished following Suzuki in [Suz1]. Here $\overline{\chi}$ is an exceptional genuine character (see Definition 2.1) of the center $Z(\overline{T})$ of the covering torus $\overline{T} \subseteq \overline{G}$. The reason for considering this problem is two-fold.

First, $\Theta(\overline{G}^{(n)}, \overline{\chi})$ is in certain sense the simplest family of genuine representations of a general covering group $\overline{G}^{(n)}$. Indeed, if $n = 1$, then it follows from definition that $\Theta(\overline{G}^{(n)}, \overline{\chi})$ could be the trivial representation of the linear group $\overline{G} = \overline{G}^{(1)}$, depending on a proper choice of the exceptional character $\overline{\chi}$. Therefore, for the genericity question regarding Whittaker functionals of genuine representations, it is reasonable to consider this family first. Moreover, theta representations for the Kazhdan-Patterson covering groups of GL_r , to which we have just alluded, are already studied in depth in the seminar paper [KP]. Despite the fact that the idea in [KP] could be applicable for general covering groups, to the best of our knowledge, it seems that there is no systematic treatment on theta representations for general covering groups in literature. Perhaps this gap is caused by the tedious cocycle computation to be carried out by any potential author. However, the Brylinski-Deligne framework enables us to compute by invoking some neat structural fact of the covering groups of interest, and to handle only a minimized usage of cocycle on the torus. In brief, we wish to fill in the gap by generalizing the relevant work of Kazhdan-Patterson to Brylinski-Deligne covering groups.

Second, distinguished theta representations have important and emergingly wider applications. Theta representations are the representation-theoretic analogues of theta functions, one of the early applications of which was given by Riemann in his seminar paper to prove the functional equation of the Riemann zeta function. In the language of modern theory of representations, theta representations gain deep applications in the Shimura correspondence (cf. [Shi] [Gel]). Following the work of Kazhdan-Patterson, theta representations on $\overline{\mathrm{GL}}_r^{(n)}$ are also studied extensively in the work Bump-Hoffstein (cf. [BH]) and Suzuki ([Suz1], [Suz2]), to mention a few. In particular, these authors made some deep conjectures and also provided evidence for a generalized Shimura correspondence regarding $\overline{\mathrm{GL}}_r^{(n)}$, and the distinguishedness property is exploited to achieve the goals in their work. Another significant direction of applications

is the Rankin-Selberg integral representation for the symmetric square and cube L -functions (cf. [BG], [BGH], [Tak] and [Ka2]). Evidently, it should be mentioned that for distinguished theta representations, the theory of L -function could be developed as in the linear algebraic case, since the Casselman-Shalika formula is then scalar-valued. More recently, the work of E. Kaplan [Ka1]-[Ka3], S. Friedberg and D. Ginzburg [FG1]-[FG3] also relies heavily on the local and global theta representations in their consideration of Fourier coefficient, Rankin-Selberg L -function and descent integral etc. Notably in their work, distinguishedness is responsible for proving that a global integral admits an Euler factorization into local factors. Besides these, the problem on global cuspidal theta representations is important and many problems are open (cf. [FG1], [Suz1]). In any case, we believe that distinguished theta representations are objects of great interest and significance, and we hope that our paper could shed some light on the relevant questions.

1.2. Main results. Generally, for fixed parity of n , the dimension $\dim \text{Wh}_\psi(\Theta(\overline{G}^{(n)}, \overline{\chi}))$ of ψ -Whittaker functionals of $\Theta(\overline{G}^{(n)}, \overline{\chi})$ increases as n increases, and a priori it depends on the tuple $(\overline{G}^{(n)}, \overline{\chi})$, in particular on a choice of the exceptional character χ . Here the dependence on $\overline{G}^{(n)}$ is a manifestation of the dependence on the root data of G , the Brylinski-Deligne classification data of central extensions (see §2) and n . Thus, the pair $(\overline{G}^{(n)}, \overline{\chi})$ such that $\dim \text{Wh}_\psi(\Theta(\overline{G}^{(n)}, \overline{\chi})) = 1$ is quite unique, and the goal is to determine completely the pair $(\overline{G}^{(n)}, \overline{\chi})$ such that $\Theta(\overline{G}^{(n)}, \overline{\chi})$ is distinguished. We remark that for fixed $\overline{G}^{(n)}$, the set of unramified exceptional characters $\overline{\chi}$ is a torsor over $Z(\overline{G}^\vee)$, the center of the complex dual group \overline{G}^\vee of \overline{G} . For details on \overline{G}^\vee , we refer to [FL], [Mc1], [Re] and [We2].

We outline the structure of the paper and state the main results.

In §2, we recall the basic structural facts on a Brylinski-Deligne covering group $\overline{G}^{(n)}$ which will be crucial for our computations. In this paper, we consider exclusively unramified covering group $\overline{G}^{(n)}$ and unramified exceptional character $\overline{\chi}$. In §3, the space $\text{Wh}_\psi(\Theta(\overline{G}^{(n)}, \overline{\chi}))$ is analyzed following the strategy in [KP] closely. In particular, it relies crucially on the rank-one local Shahidi coefficient matrix $[\tau(\overline{\chi}, \mathbf{w}_\alpha, \gamma, \gamma')]_{\gamma, \gamma'}$ for covering groups. In the unramified setting, the matrix is computed in [Mc2]; it is also computed for ramified places in [GS]. The first main result is Theorem 3.14 from §3:

Theorem 1.1. *Let $\overline{G}^{(n)}$ be an arbitrary unramified Brylinski-Deligne covering group. Let $\overline{\chi}$ be an unramified exceptional genuine character of $\overline{G}^{(n)}$ with associated theta representation $\Theta(\overline{G}^{(n)}, \overline{\chi})$. Then,*

$$\left| \wp_{Q,n}(\mathcal{O}_{Q,n}^F) \right| \leq \dim \text{Wh}_\psi(\Theta(\overline{G}^{(n)}, \overline{\chi})) \leq \left| \wp_{Q,n}(\mathcal{O}_{Q,n,\text{sc}}^F) \right|.$$

The notations are referred to §2.

Note that for covering groups $\overline{\text{GL}}_r^{(n)}$, theta representations are studied by Kazhdan-Patterson extensively in [KP]. In particular, they determine $\dim \text{Wh}_\psi(\Theta(\overline{\text{GL}}_r^{(n)}, \overline{\chi})) = 1$ if and only if

- 1) $n = r$ and $\overline{\text{GL}}_r^{(n)}$ is any Kazhdan-Patterson covering group, or
- 2) $n = r + 1$ and $\overline{\text{GL}}_r^{(n)}$ belongs to a special type of degree n Kazhdan-Patterson covering groups.

Our Theorem 1.1 above recovers their result. More precisely, for any covering group $\overline{\text{GL}}_r^{(n)}$ studied in [KP], one has $\mathcal{O}_{Q,n}^F = \mathcal{O}_{Q,n,\text{sc}}^F$. Therefore $\dim \text{Wh}_\psi(\Theta(\overline{\text{GL}}_r^{(n)}, \overline{\chi})) = \left| \wp_{Q,n}(\mathcal{O}_{Q,n,\text{sc}}^F) \right|$, and in particular it does not depend on the choice of the exceptional character $\overline{\chi}$. For details, see Example 3.16.

In general, for cases where the two bounds in Theorem 1.1 actually agree, the computation of the dimension is reduced to purely combinatorial problem, and thus amenable to straightforward calculation. This includes the case where $Y_{Q,n} = Y_{Q,n}^{\text{sc}}$, or equivalently $Z(\overline{G}^\vee) = 1$. For example, odd degree coverings of simply-connected groups of type B_r, C_r have this property. See §5, §6.

In contrast, when the two bounds in Theorem 1.1 do not agree, $\dim \text{Wh}_\psi(\Theta(\overline{G}, \overline{\chi}))$ becomes sensitive to the choice of the exceptional character $\overline{\chi}$. The second half of this paper is devoted to investigate this. Note that this phenomenon already occurs for the degree two metaplectic covering $\overline{\text{SL}}_2^{(2)}$, see Example 4.6. In this case $\Theta(\overline{\text{SL}}_2^{(2)}, \overline{\chi})$ is the even Weil representation. Consider $\Theta(\overline{\text{SL}}_2^{(2)}, \overline{\chi}_{\psi_a})$, where $\overline{\chi}_{\psi_a}$ is an exceptional character defined by using the twisted additive character $\psi_a, a \in F^\times$. It is well-known that $\dim \text{Wh}_\psi(\Theta(\overline{\text{SL}}_2^{(2)}, \overline{\chi}_{\psi_a})) \leq 1$ and the equality holds if and only if $a \in (F^\times)^2$. Our analysis shows that similar phenomenon occurs for higher rank groups, see §4.2, in particular Corollary 4.5.

In any case, we summarize our results for certain coverings of simply-connected groups as follows. We write for instance $\overline{A}_r^{(n)}$ for the degree n covering of the simply-connected group of type A_r of rank r . Here the covering group arises from a quadratic form Q on the coroot lattice $Y = Y^{\text{sc}}$ such that $Q(\alpha^\vee) = 1$ for any short coroot α^\vee . The following theorem is an amalgam of Theorem 4.9, Theorem 5.3, Theorem 6.2 and Theorem 7.1. Only for $\overline{A}_r^{(n)}$, we impose the condition $n \leq r + 2$ for simplicity purpose.

Theorem 1.2. *Let $\overline{G}^{(n)}$ be an unramified Brylinski-Deligne degree n covering of a simply-connected semisimple group of type A_r, B_r, C_r or G_2 . If $\overline{G}^{(n)} = \overline{A}_r^{(n)}$, we further assume $n \leq r + 2$. Let $\overline{\chi}$ be an unramified exceptional character for $\overline{G}^{(n)}$. In each case for $\overline{G}^{(n)}$ below, if $\dim \text{Wh}_\psi(\Theta(\overline{G}^{(n)}, \overline{\chi})) = 1$, then necessarily the following relation between r and n holds:*

$$\begin{cases} \overline{A}_r^{(n)}, r \geq 1, n \leq r + 2 : & n = r + 2 \text{ or } r + 1; \\ \overline{C}_r^{(n)}, r \geq 2 : & n = 4r - 2, 4r, 4r + 2 \text{ or } 2r + 1; \\ \overline{B}_r^{(n)}, r \geq 3 : & n = 2r + 1 \text{ or } 2r + 2; \\ \overline{G}_2^{(n)} : & n = 7 \text{ or } 12. \end{cases}$$

Conversely, suppose r and n satisfy the above relations; then for every case above except $\overline{C}_r^{(4r)}$, there exists a unique exceptional character $\overline{\chi}$ such that $\dim \text{Wh}_\psi(\Theta(\overline{G}^{(n)}, \overline{\chi})) = 1$ for above $\overline{G}^{(n)}$.

We actually determine the unique exceptional character specified in Theorem 1.2, see Theorems 4.9, 5.3, 6.2 and 7.1. In the $\overline{A}_r^{(r+1)}$ case, our result generalizes that for the even Weil representation of $\overline{\text{SL}}_2^{(2)}$ mentioned above. As noted, the collection of unramified exceptional characters is a torsor over $Z(\overline{G}^\vee)$. However, for covering groups of simply connected groups, the choice of ψ actually gives a base point for this torsor. Thus, any exceptional character $\overline{\chi}$ gives rise to an element in $Z(\overline{G}^\vee)$, depending on the choice of ψ . That is, the explicit requirement given in those theorems could be viewed as determining the corresponding element in $Z(\overline{G}^\vee)$.

We note that for classical groups and similitude groups, an extensive study is included in [FGS]. Our result from Theorem 1.2 also agrees with the pertinent discussion in [FG2] for symplectic groups. For example, the local statement for the second part of Conjecture 1 in [FG2] follows from Proposition 5.1 in our paper. Moreover, the factorizability property of the Whittaker function in [FG2] for $\overline{\text{Sp}}_{2n}^{(4n-2)}$ also agrees with our result for the $\overline{C}_r^{(n)}$ case in Theorem 1.2.

Finally, we remark that groups of type D_r, E_6, E_7, E_8, F_4 could be analyzed by the same procedure. In principle, Theorem 1.1 coupled with analogous argument for Theorem 1.2 enable one to determine completely $\dim \text{Wh}_\psi(\Theta(\overline{G}^{(n)}, \overline{\chi}))$ for arbitrary $(\overline{G}^{(n)}, \overline{\chi})$.

Acknowledgment. I would like to thank Solomon Friedberg, David Goldberg and Freydoon Shahidi for interesting discussions on various topics. Thanks are also due to Wee Teck Gan for helpful comments on the paper. Meanwhile, I would like to thank Boston College for its support and hospitality during a visit in the Fall semester.

2. BASIC SET-UPS

2.1. Structural facts on \overline{G} . To ease the reading for the reader, we recall first some structural facts on \overline{G} . The main references are [BD], [FL], [Re], [Mc1]-[Mc2], [We2] and [GG]. In this paper, we concentrate exclusively on unramified Brylinski-Deligne covering group \overline{G} (to be explained below). We follow notations in [GG].

Let F be a nonarchimedean field of characteristic 0, with residual characteristic p . Fix a uniformiser ϖ of F . Let \mathbb{G} be a split linear algebraic group over F with maximal split torus \mathbb{T} . Write $(X, \Phi, \Delta, Y, \Phi^\vee, \Delta^\vee)$ for the root data of \mathbb{G} . Here X (resp. Y) is the character lattice (resp. cocharacter lattice) for (\mathbb{G}, \mathbb{T}) . Choose a set $\Delta \subseteq \Phi$ of simple roots from the set of roots Φ , and Δ^\vee the corresponding simple coroots from Φ^\vee . Let \mathbb{B} be the Borel subgroup associated with Δ . Write $Y^{\text{sc}} \subseteq Y$ for the lattice generated by Φ^\vee .

Fix a Chevalley system of pinnings for $(\mathbb{G}, \mathbb{T}, \mathbb{B})$. That is, fix an isomorphism $e_\alpha : \mathbb{G}_a \rightarrow \mathbb{U}_\alpha$ for each $\alpha \in \Phi$, where $\mathbb{U}_\alpha \subseteq \mathbb{G}$ is the root subgroup associated with α . Moreover, for each $\alpha \in \Phi$, there is the induced morphism $\varphi_\alpha : \text{SL}_2 \rightarrow \mathbb{G}$ which restricts to $e_{\pm\alpha}$ on the upper and lower triangular subgroup of unipotent matrices of SL_2 .

Consider the algebro-geometric covering $\overline{\mathbb{G}}$ of \mathbb{G} by \mathbb{K}_2 , which is categorically equivalent to the pairs $\{(D, \eta)\}$ (cf. [GG]). Here $\eta : Y^{\text{sc}} \rightarrow F^\times$ is a homomorphism. On the other hand, D is a bisector associated to a Weyl-invariant quadratic form $Q : Y \rightarrow \mathbb{Z}$. That is, let B_Q be the Weyl-invariant bilinear form associated to Q such that $B_Q(y_1, y_2) = Q(y_1 + y_2) - Q(y_1) - Q(y_2)$, then D is a bilinear form on Y satisfying

$$D(y_1, y_2) + D(y_2, y_1) = B_Q(y_1, y_2).$$

The bisector D is not necessarily symmetric. Any $\overline{\mathbb{G}}$ is, up to isomorphism, incarnated by (i.e. categorically associated to) (D, η) for a bisector D and some η .

Let $n \geq 1$ be a natural number. Assume F^\times contains the full group μ_n of n -th roots of unity and $p \nmid n$. Let $\overline{\mathbb{G}}$ be incarnated by (D, η) . One obtains naturally degree n topological covering groups $\overline{G}, \overline{T}, \overline{B}$ of the rational points $G := \mathbb{G}(F), T := \mathbb{T}(F), B := \mathbb{B}(F)$, e.g.,

$$\mu_n \hookrightarrow \overline{G} \twoheadrightarrow G.$$

We may write $\overline{G}^{(n)}$ for \overline{G} to emphasize the degree of covering. For any set $H \subseteq G$, we write $\overline{H} \subseteq \overline{G}$ for the preimage of H with respect to the quotient map $\overline{G} \rightarrow G$. The Bruhat-Tits theory gives a maximal compact subgroup $K \subseteq G$, which depends on the fixed pinnings. We assume that \overline{G} splits over K and fix such a splitting; call \overline{G} an unramified Brylinski-Deligne covering group in this case. The data (D, η) play the following role for the structural fact on \overline{G} .

- The group \overline{G} splits canonically over any unipotent element of G . In particular, we write $\overline{e}_\alpha(u) \in \overline{G}, \alpha \in \Phi, u \in F$ for the canonical lifting of $e_\alpha(u) \in G$. For any $\alpha \in \Phi$, there is a natural representative $w_\alpha := e_\alpha(1)e_{-\alpha}(1)e_\alpha(1) \in K$ (and therefore $\overline{w}_\alpha \in \overline{G}$ by the splitting of K) of the Weyl element $w_\alpha \in W$. Moreover, for $h_\alpha(a) := \alpha^\vee(a) \in G, \alpha \in \Phi, a \in F^\times$, there is a natural lifting $\overline{h}_\alpha(a) \in \overline{G}$ of $h_\alpha(a)$, which depends only on the pinning and the canonical unipotent splitting. For details, see [GG].
- There is a section \mathbf{s} of T into \overline{T} such that the group law on \overline{T} is given by

$$(1) \quad \mathbf{s}(y_1(a)) \cdot \mathbf{s}(y_2(b)) = (a, b)_n^{D(y_1, y_2)} \cdot \mathbf{s}(y_1(a) \cdot y_2(b)).$$

Moreover, for the natural lifting $\overline{h}_\alpha(a)$, one has

$$(2) \quad \overline{h}_\alpha(a) = (\eta(\alpha^\vee), a)_n \cdot \mathbf{s}(h_\alpha(a)) \in \overline{T}.$$

- Let $w_\alpha \in G$ be the natural representative of $w_\alpha \in W$. For any $\overline{y(a)} \in \overline{T}$, one has

$$(3) \quad w_\alpha \cdot \overline{y(a)} \cdot w_\alpha^{-1} = \overline{y(a)} \cdot \overline{h}_\alpha(a^{-\langle y, \alpha \rangle}),$$

where $\langle -, - \rangle$ is the paring between Y and X .

Consider the sublattice $Y_{Q,n} := \{y \in Y : B_Q(y, y') \in n\mathbb{Z}\}$ of Y . For every $\alpha^\vee \in \Phi^\vee$, define $n_\alpha := n/\text{gcd}(n, Q(\alpha^\vee))$. Write $\alpha_{Q,n}^\vee := n_\alpha \alpha^\vee$ and $\alpha_{Q,n} := n_\alpha^{-1} \alpha$. Let $Y_{Q,n}^{\text{sc}} \subseteq Y$ be the sublattice

generated by $\{\alpha_{Q,n}^\vee\}_{\alpha \in \Phi}$. The complex dual group \overline{G}^\vee for \overline{G} as given in [FL], [Mc1] and [Re] has root data $(Y_{Q,n}, \{\alpha_{Q,n}^\vee\}, \text{Hom}(Y_{Q,n}, \mathbf{Z}), \{\alpha_{Q,n}\})$. In particular, $Y_{Q,n}^{\text{sc}}$ is the root lattice for \overline{G}^\vee . What is most pertinent to our paper is that the center $Z(\overline{G}^\vee)$ could be identified as

$$Z(\overline{G}^\vee) := \text{Hom}(Y_{Q,n}/Y_{Q,n}^{\text{sc}}, \mathbf{C}^\times).$$

2.2. Theta representations $\Theta(\overline{G}, \overline{\chi})$. Fix an embedding $\iota : \mu_n \hookrightarrow \mathbf{C}^\times$. A representation of \overline{G} is called ι -genuine if μ_n acts via ι . We consider throughout the paper ι -genuine (or simply genuine) representations of \overline{G} .

Let U be the unipotent subgroup of $B = TU$. As U splits canonically in \overline{G} , we have $\overline{B} = \overline{T}U$. The covering torus \overline{T} is a Heisenberg group with center $Z(\overline{T})$. The image of $Z(\overline{T})$ in T is equal to the image of the isogeny $Y_{Q,n} \otimes F^\times \rightarrow T$ induced from $Y_{Q,n} \rightarrow Y$.

Let $\overline{\chi} \in \text{Hom}_\iota(Z(\overline{T}), \mathbf{C}^\times)$ be a genuine character of $Z(\overline{T})$, write $i(\overline{\chi}) := \text{Ind}_A^{\overline{T}} \overline{\chi}'$ for the induced representation on \overline{T} , where A is any maximal abelian subgroup of \overline{T} containing $Z(\overline{T})$, and $\overline{\chi}'$ is any extension of $\overline{\chi}$. By the Stone von-Neumann theorem (cf. [We1, Theorem 3.1] and [Mc1, Theorem 3]), $\overline{\chi} \mapsto i(\overline{\chi})$ gives a bijection between isomorphism classes of genuine representations of $Z(\overline{T})$ and \overline{T} . For unramified group \overline{G} , we take \overline{A} to be $Z(\overline{T}) \cdot (K \cap T)$ from now.

View $i(\overline{\chi})$ as a genuine representation of \overline{B} by inflation from the quotient map $\overline{B} \rightarrow \overline{T}$. Write $I(i(\overline{\chi})) := \text{Ind}_{\overline{B}}^{\overline{G}} i(\overline{\chi})$ for the normalized induced principal series representation of \overline{G} . For simplicity, we may also write $I(\overline{\chi})$ for $I(i(\overline{\chi}))$. One knows that $I(\overline{\chi})$ is unramified (i.e. $I(\overline{\chi})^K \neq 0$) if and only if $\overline{\chi}$ is unramified, i.e., $\overline{\chi}$ is trivial on $Z(\overline{T}) \cap K$. We consider in this paper only unramified genuine representations (and characters). In fact, one has the naturally arising abelian extension

$$(4) \quad \mu_n \hookrightarrow \overline{Y}_{Q,n} \twoheadrightarrow Y_{Q,n}$$

such that unramified genuine characters of $\overline{\chi}$ of $Z(\overline{T})$ correspond to genuine characters of $\overline{Y}_{Q,n}$. Here $\overline{Y}_{Q,n} := Z(\overline{T})/Z(\overline{T}) \cap K$. As mentioned, for unramified group \overline{G} , we take \overline{A} to be $Z(\overline{T}) \cdot (K \cap T)$. Then there is a canonical extension (also denoted by $\overline{\chi}$) of an unramified character $\overline{\chi}$ of $Z(\overline{T})$ to \overline{A} , since $\overline{A}/(T \cap K) \simeq \overline{Y}_{Q,n}$. Therefore, we will identify $i(\overline{\chi})$ as $\text{Ind}_{\overline{A}}^{\overline{T}} \overline{\chi}$ for this \overline{A} and $\overline{\chi}$.

If $I(\overline{\chi})$ is unramified, then for any $\alpha \in \Delta$, the intertwining operator $T_{\mathbf{w}_\alpha, \overline{\chi}} : I(\overline{\chi}) \rightarrow I({}^{\mathbf{w}_\alpha} \overline{\chi})$ is determined by (cf. [Mc2, Theorem 12.1] and [Gao])

$$T_{\mathbf{w}_\alpha, \overline{\chi}}(f_0) = c(\mathbf{w}_\alpha, \overline{\chi}) \cdot f'_0 \text{ with } c(\mathbf{w}_\alpha, \overline{\chi}) = \frac{1 - q^{-1} \overline{\chi}(\overline{h}_\alpha(\varpi^{n_\alpha}))}{1 - \overline{\chi}(\overline{h}_\alpha(\varpi^{n_\alpha}))},$$

where $f_0 \in I(\overline{\chi})$ and $f'_0 \in I({}^{\mathbf{w}_\alpha} \overline{\chi})$ are the unramified vectors.

The following definition mimic that in [KP, §I.2].

Definition 2.1. An unramified genuine character $\overline{\chi}$ of $Z(\overline{T})$ is called exceptional if $\overline{\chi}(\overline{h}_\alpha(\varpi^{n_\alpha})) = q^{-1}$ for all $\alpha \in \Delta$. The theta representation $\Theta(\overline{G}, \overline{\chi})$ associated to an exceptional character $\overline{\chi}$ is the unique Langlands quotient (cf. [BJ]) of $I(\overline{\chi})$, which is also equal to the image of the intertwining operator $T_{\mathbf{w}_0, \overline{\chi}} : I(\overline{\chi}) \rightarrow I({}^{\mathbf{w}_0} \overline{\chi})$, where $\mathbf{w}_0 \in W$ is the longest Weyl element.

The extension $\overline{Y}_{Q,n}$ gives rise to an extension $\overline{Y}_{Q,n}^{\text{sc}}$ of $Y_{Q,n}^{\text{sc}}$ by restriction. All exceptional characters agree on $\overline{Y}_{Q,n}^{\text{sc}}$, and therefore the set of exceptional characters is a torsor over $Z(\overline{G}^\vee)$.

2.3. Unitary distinguished characters. A special class of the so-called distinguished genuine characters of $Z(\overline{T})$ is singled out in [GG] for the consideration of the L -group extension for \overline{G} . Distinguished characters, in the sense of [GG], may not exist for general Brylinski-Deligne covering groups. However, if \mathbb{G} has simply-connected derived group or if the composition $\eta : Y^{\text{sc}} \rightarrow F^\times \rightarrow F/(F^\times)^n$ is trivial, there exist such characters.

For the purpose of §4-§7, we recall the explicit construction in [GG] when a distinguished character exists. In particular, we make the above assumption on \overline{G} , which is clearly satisfied in the simply-connected case in §4-§7.

First, let $\{y_i\}$ be a basis of $Y_{Q,n}$ such that $\{k_i y_i\}$ is a basis for the lattice $J = nY + Y_{Q,n}^{\text{sc}}$ for some $k_i \in \mathbf{Z}$. Let ψ' be a nontrivial additive character of F . Let $\gamma_{\psi'}$ be the Weil index. Then by definition a unitary distinguished character $\overline{\chi}_{\psi'}^0$ of $Z(\overline{T})$ is given by

$$\overline{\chi}_{\psi'}^0(y_i(a)) = \gamma_{\psi'}(a)^{\frac{2(k_i-1)Q(y_i)}{n}},$$

and for $y = \sum_i n_i y_i$ and $a \in F^\times$,

$$(5) \quad \overline{\chi}_{\psi'}^0(y(a)) = (a, a)_n^{\sum_{i < j} n_i n_j D(y_i, y_j)} \cdot \prod_i \overline{\chi}_{\psi'}^0(y_i(a^{n_i}))^{\frac{2(k_i-1)Q(y_i)}{n}}.$$

Note in [GG], the exponent of $\gamma_{\psi'}(a)$ in the formula of $\overline{\chi}_{\psi'}^0(y_i(a))$ is the negative of what we use here. However, both give rise to distinguished characters.

2.4. Conventions and notations. Let $2\rho := \sum_{\alpha^\vee > 0} \alpha^\vee$ be the sum of all positive coroots of \mathbb{G} . Consider the affine translation $\ell_\rho : Y \otimes \mathbf{Q} \rightarrow Y \otimes \mathbf{Q}$ given by $y \mapsto y - \rho$. Write $\mathfrak{w}(y)$ for the natural Weyl group action on Y and $Y \otimes \mathbf{Q}$. Endow the codomain of ℓ_ρ with this action. By transport of structure, one has an induced action of W on the domain of ℓ_ρ (i.e. the first $Y \otimes \mathbf{Q}$), which we denote by $\mathfrak{w}[y]$. That is,

$$\mathfrak{w}[y] := \mathfrak{w}(y - \rho) + \rho.$$

Clearly Y is stable under this action. Write $y_\rho := y - \rho$ for any $y \in Y$, then $\mathfrak{w}[y] - y = \mathfrak{w}(y_\rho) - y_\rho$. From now, by Weyl orbits in Y or $Y \otimes \mathbf{Q}$ we always refer to the ones with respect to the action $\mathfrak{w}[y]$. Write \mathcal{O} (respectively \mathcal{O}^F) for the set of W -orbits (resp. free W -orbits) in Y .

Definition 2.2. For any subgroup $\Lambda \subseteq Y$, a free orbit $\mathcal{O}_y \in \mathcal{O}^F$ is called Λ -free if the quotient map $Y \rightarrow Y/\Lambda$ is injective on \mathcal{O}_y . We write $\mathcal{O}_\Lambda^F \subseteq \mathcal{O}^F$ for the set of Λ -free orbits of Y .

Note that Λ -free orbits are assumed to be free by definition. For simplicity, we will write $\mathcal{O}_{Q,n,\text{sc}}^F$ and $\mathcal{O}_{Q,n}^F$ for the set of $Y_{Q,n}^{\text{sc}}$ and $Y_{Q,n}$ -free orbits of Y respectively. Clearly, the inclusions $\mathcal{O} \supseteq \mathcal{O}^F \supseteq \mathcal{O}_{Q,n,\text{sc}}^F \supseteq \mathcal{O}_{Q,n}^F$ hold.

Generally, notations will be either self-explanatory or explained the first time they occur. For convenience, we list some notations which appear frequently in the text:

ε : the element $\iota((-1, \varpi)_n) \in \mathbf{C}^\times$. In particular, for n odd, $\varepsilon = 1$. We use freely in the paper the following identity:

$$\varepsilon^{D(y, y')} = \varepsilon^{D(y', y)} \text{ for any } y \in Y_{Q,n}, y' \in Y.$$

$\wp_{Q,n}$: the projection $Y \rightarrow Y/Y_{Q,n}$.

$\wp_{Q,n}^{\text{sc}}$: the projection $Y \rightarrow Y/Y_{Q,n}^{\text{sc}}$.

ψ : a fixed additive character of F into \mathbf{C}^\times with conductor O_F . For any $a \in F^\times$, the twisted character ψ_a is given by $\psi_a : x \mapsto \psi(ax)$.

\mathbf{s}_y : for any $y \in Y$, we write $\mathbf{s}_y := \mathbf{s}(\varpi^y) \in \overline{T}$.

$\gamma_{\psi'}$: the Weil index attached to a nontrivial character $\psi' : F \rightarrow \mathbf{C}^\times$.

$[x]$: the minimum integer such that $[x] \geq x$ for a real number x .

3. BOUNDS FOR $\dim \text{Wh}_\psi(\Theta(\overline{G}, \overline{\chi}))$

3.1. Whittaker functionals. Let \overline{G} be an unramified Brylinski-Deligne n -fold covering group with Borel subgroup $\overline{B} = \overline{T}U$. Consider the principal series $I(\overline{\chi}) := I(i(\overline{\chi}))$ for an unramified character $\overline{\chi} \in \text{Hom}_\iota(Z(\overline{T}), \mathbf{C}^\times)$. We follow the notations in §2.2.

Let $\text{Ftn}(i(\overline{\chi}))$ be the vector space of functions \mathbf{c} on \overline{T} satisfying

$$\mathbf{c}(\overline{t} \cdot \overline{z}) = \mathbf{c}(\overline{t}) \cdot \overline{\chi}(\overline{z}), \quad \overline{t} \in \overline{T} \text{ and } \overline{z} \in \overline{A}.$$

The support of any $\mathbf{c} \in \mathbf{Ftn}(i(\overline{\chi}))$ is a disjoint union of cosets in $\overline{T}/\overline{A}$. Moreover, $\dim \mathbf{Ftn}(i(\overline{\chi})) = |Y/Y_{Q,n}|$ since $\overline{T}/\overline{A}$ has the same size as $Y/Y_{Q,n}$.

There is a natural isomorphism of vector spaces $\mathbf{Ftn}(i(\overline{\chi})) \simeq i(\overline{\chi})^\vee$, where $i(\overline{\chi})^\vee$ is the complex dual space of functionals of $i(\overline{\chi})$. More explicitly, let $\{\gamma_i\} \subseteq \overline{T}$ be a chosen set of representatives of $\overline{T}/\overline{A}$, consider $\mathbf{c}_{\gamma_i} \in \mathbf{Ftn}(i(\overline{\chi}))$ which has support $\gamma_i \cdot \overline{A}$ and $\mathbf{c}_{\gamma_i}(\gamma_i) = 1$. It gives rise to a linear functional $\lambda_{\gamma_i}^\vee \in i(\overline{\chi})^\vee$ such that $\lambda_{\gamma_i}^\vee(f_{\gamma_j}) = \delta_{ij}$, where $f_{\gamma_j} \in i(\overline{\chi})$ is the unique element such that $\text{supp}(f_{\gamma_j}) = \overline{A} \cdot \gamma_j^{-1}$ and $f_{\gamma_j}(\gamma_j^{-1}) = 1$. That is, $f_{\gamma_j} = i(\overline{\chi})(\gamma_j)\phi_0$, where $\phi_0 \in i(\overline{\chi})$ is the normalized unramified vector of $i(\overline{\chi})$ such that $\phi_0(1_{\overline{T}}) = 1$. Thus, the isomorphism $\mathbf{Ftn}(i(\overline{\chi})) \simeq i(\overline{\chi})^\vee$ is given explicitly by

$$\mathbf{c} \mapsto \lambda_{\mathbf{c}}^\vee := \sum_{\gamma_i \in \overline{T}/\overline{A}} \mathbf{c}(\gamma_i) \lambda_{\gamma_i}^\vee.$$

It can be checked easily that the isomorphism does not depend on the choice of representatives for $\overline{T}/\overline{A}$.

Fix an additive character $\psi : F \rightarrow \mathbf{C}^\times$ of conductor O_F . Let $\psi_U : U \rightarrow \mathbf{C}^\times$ be the character on U such that its restriction to every $U_\alpha, \alpha \in \Delta$ is given by $\psi \circ e_\alpha^{-1}$.

Definition 3.1. For any genuine representation $(\overline{\sigma}, V_{\overline{\sigma}})$ of \overline{G} , a linear functional $\ell : V_{\overline{\sigma}} \rightarrow \mathbf{C}$ is called a Whittaker functional if $\ell(\overline{\sigma}(u)v) = \psi_U(u) \cdot v$ for all $u \in U$ and $v \in V_{\overline{\sigma}}$. Write $\mathbf{Wh}_\psi(\overline{\sigma})$ for the space of Whittaker functionals for $\overline{\sigma}$.

There is an isomorphism between $i(\overline{\chi})^\vee$ and the space $\mathbf{Wh}_\psi(I(\overline{\chi}))$ of Whittaker functionals on $I(\overline{\chi})$ (cf. [Mc2]), given by $\lambda \mapsto W_\lambda$ with

$$W_\lambda : I(\overline{\chi}) \rightarrow \mathbf{C}, \quad f \mapsto \lambda \left(\int_{U^-} f(uw_0) \psi(u) \mu(u) \right),$$

where $f \in I(\overline{\chi})$ is an $i(\overline{\chi})$ -valued function on \overline{G} . Here U^- is the unipotent subgroup opposite to the unipotent radical $U \subseteq B$; also, $w_0 = w_{\alpha_1} w_{\alpha_2} \dots w_{\alpha_k} \in K$ is a representative of \mathfrak{w}_0 , where $\mathfrak{w}_0 = \mathfrak{w}_{\alpha_1} \mathfrak{w}_{\alpha_2} \dots \mathfrak{w}_{\alpha_k}$ is a minimum decomposition of \mathfrak{w}_0 . For any $\mathbf{c} \in \mathbf{Ftn}(i(\overline{\chi}))$, by abuse of notation, we will write $\lambda_{\mathbf{c}}^\vee \in \mathbf{Wh}_\psi(I(\overline{\chi}))$ for the resulting Whittaker functional of $I(\overline{\chi})$ from the isomorphism $\mathbf{Ftn}(i(\overline{\chi})) \simeq i(\overline{\chi})^\vee \simeq \mathbf{Wh}_\psi(I(\overline{\chi}))$. An easy consequence is

$$\dim \mathbf{Wh}_\psi(I(\overline{\chi})) = |Y/Y_{Q,n}|.$$

Let $T_{\mathfrak{w}, \overline{\chi}} : I(\overline{\chi}) \rightarrow I({}^{\mathfrak{w}}\overline{\chi})$, $\mathfrak{w} \in W$ be the intertwining operator for $\mathfrak{w} \in W$. Let $J(\mathfrak{w}, \overline{\chi})$ be the image of $T_{\mathfrak{w}, \overline{\chi}}$. The operator $T_{\mathfrak{w}, \overline{\chi}}$ induces a homomorphism $T_{\mathfrak{w}, \overline{\chi}}^*$ of vector spaces with image $\mathbf{Wh}_\psi(J(\mathfrak{w}, \overline{\chi}))$:

$$\begin{array}{ccc} T_{\mathfrak{w}, \overline{\chi}}^* : \mathbf{Wh}_\psi(I({}^{\mathfrak{w}}\overline{\chi})) & \longrightarrow & \mathbf{Wh}_\psi(I(\overline{\chi})) \\ & \searrow & \uparrow \\ & & \mathbf{Wh}_\psi(J(\mathfrak{w}, \overline{\chi})), \end{array}$$

which is given by $\langle \lambda_{\mathbf{c}}^{\mathfrak{w}\overline{\chi}}, - \rangle \mapsto \langle \lambda_{\mathbf{c}}^{\mathfrak{w}\overline{\chi}}, T_{\mathfrak{w}, \overline{\chi}}(-) \rangle$ for any $\mathbf{c} \in \mathbf{Ftn}(i({}^{\mathfrak{w}}\overline{\chi}))$. Let $\{\lambda_{\gamma}^{\mathfrak{w}\overline{\chi}}\}_{\gamma \in \overline{T}/\overline{A}}$ be a basis for $\mathbf{Wh}_\psi(I({}^{\mathfrak{w}}\overline{\chi}))$, and $\{\lambda_{\gamma'}^{\overline{\chi}}\}$ a basis for $\mathbf{Wh}_\psi(I(\overline{\chi}))$. The map $T_{\mathfrak{w}, \overline{\chi}}^*$ is then determined by the matrix $[\tau(\overline{\chi}, \mathfrak{w}, \gamma, \gamma')]_{\gamma, \gamma' \in \overline{T}/\overline{A}}$ of size $|Y/Y_{Q,n}|$ such that

$$T_{\mathfrak{w}, \overline{\chi}}^*(\lambda_{\gamma}^{\mathfrak{w}\overline{\chi}}) = \sum_{\gamma' \in \overline{T}/\overline{A}} \tau(\overline{\chi}, \mathfrak{w}, \gamma, \gamma') \cdot \lambda_{\gamma'}^{\overline{\chi}}.$$

Some immediate properties are

Lemma 3.2. For $\mathfrak{w} \in W$ and $\overline{z}, \overline{z}' \in \overline{A}$, the identity holds:

$$\tau(\overline{\chi}, \mathfrak{w}, \gamma \cdot \overline{z}, \gamma' \cdot \overline{z}') = ({}^{\mathfrak{w}}\overline{\chi})^{-1}(\overline{z}) \cdot \tau(\overline{\chi}, \mathfrak{w}, \gamma, \gamma') \cdot \overline{\chi}(\overline{z}').$$

Moreover, for $\mathbf{w}_1, \mathbf{w}_2 \in W$ such that $l(\mathbf{w}_2 \mathbf{w}_1) = l(\mathbf{w}_2) + l(\mathbf{w}_1)$, one has

$$\tau(\overline{\chi}, \mathbf{w}_2 \mathbf{w}_1, \gamma, \gamma') = \sum_{\gamma'' \in \overline{T}/\overline{A}} \tau(\mathbf{w}_1 \overline{\chi}, \mathbf{w}_2, \gamma, \gamma'') \cdot \tau(\overline{\chi}, \mathbf{w}_1, \gamma'', \gamma'),$$

which is referred to as the cocycle relation.

3.2. Reduction of $\text{Wh}_\psi(\Theta(\overline{G}, \overline{\chi}))$. Let \mathbf{w}_0 be the longest Weyl element of \mathbb{G} . Consider the theta representation $\Theta(\overline{G}, \overline{\chi}) = T_{\mathbf{w}_0, \overline{\chi}}(I(\overline{\chi}))$ attached to an unramified exceptional character $\overline{\chi}$ (see Definition 2.1).

Definition 3.3. A theta representation $\Theta(\overline{G}, \overline{\chi})$ attached to an unramified exceptional genuine character $\overline{\chi}$ is called distinguished if and only if $\dim \text{Wh}_\psi(\Theta(\overline{G}, \overline{\chi})) = 1$.

Proposition 3.4. Let $\overline{\chi}$ be an unramified exceptional character of \overline{G} , and Δ the set of simple roots. Then

$$\text{Wh}_\psi(\Theta(\overline{G}, \overline{\chi})) = \bigcap_{\alpha \in \Delta} \text{Ker}(T_{\mathbf{w}_\alpha, \mathbf{w}_\alpha \overline{\chi}}^* : \text{Wh}_\psi(I(\overline{\chi})) \rightarrow \text{Wh}_\psi(I(\mathbf{w}_\alpha \overline{\chi}))),$$

where $T_{\mathbf{w}_\alpha, \mathbf{w}_\alpha \overline{\chi}}$ is the intertwining operator from $I(\mathbf{w}_\alpha \overline{\chi})$ to $I(\overline{\chi})$.

Proof. The same proof for [KP, Theorem I.2.9] applies here mutatis mutandis. \square

Let $\lambda_{\overline{\chi}} \in \text{Wh}_\psi(I(\overline{\chi}))$ and $\alpha \in \Delta$, then

$$T_{\mathbf{w}_\alpha, \mathbf{w}_\alpha \overline{\chi}}^*(\lambda_{\overline{\chi}}) = \sum_{\gamma'} \tau(\mathbf{w}_\alpha \overline{\chi}, \mathbf{w}_\alpha, \gamma, \gamma') \cdot \lambda_{\gamma'}^{\mathbf{w}_\alpha \overline{\chi}}.$$

In general, let $\mathbf{c} \in \text{Ftn}(i(\overline{\chi}))$, and write

$$\lambda_{\mathbf{c}}^{\overline{\chi}} = \sum_{\gamma \in \overline{T}/\overline{A}} \mathbf{c}(\gamma) \lambda_{\gamma}^{\overline{\chi}}.$$

Then,

$$\begin{aligned} T_{\mathbf{w}_\alpha, \mathbf{w}_\alpha \overline{\chi}}^*(\lambda_{\mathbf{c}}^{\overline{\chi}}) &= \sum_{\gamma} \mathbf{c}(\gamma) \left(\sum_{\gamma'} \tau(\mathbf{w}_\alpha \overline{\chi}, \mathbf{w}_\alpha, \gamma, \gamma') \cdot \lambda_{\gamma'}^{\mathbf{w}_\alpha \overline{\chi}} \right) \\ &= \sum_{\gamma'} \left(\sum_{\gamma} \mathbf{c}(\gamma) \tau(\mathbf{w}_\alpha \overline{\chi}, \mathbf{w}_\alpha, \gamma, \gamma') \right) \lambda_{\gamma'}^{\mathbf{w}_\alpha \overline{\chi}}. \end{aligned}$$

As an immediate consequence of Proposition 3.4, one has:

Corollary 3.5. A function $\mathbf{c} \in \text{Ftn}(i(\overline{\chi}))$ gives rise to a functional in $\text{Wh}_\psi(\Theta(\overline{G}, \overline{\chi}))$ (i.e. $\lambda_{\mathbf{c}}^{\overline{\chi}} \in \text{Wh}_\psi(\Theta(\overline{G}, \overline{\chi}))$) if and only if for all $\alpha \in \Delta$,

$$\sum_{\gamma \in \overline{T}/\overline{A}} \mathbf{c}(\gamma) \tau(\mathbf{w}_\alpha \overline{\chi}, \mathbf{w}_\alpha, \gamma, \gamma') = 0 \text{ for all } \gamma'.$$

Note the left hand side is independent of the choice of representatives for $\overline{T}/\overline{A}$ by Lemma 3.2.

3.3. The Shahidi local coefficient matrix. We would like to compute the matrix $[\tau(\overline{\chi}, \mathbf{w}_\alpha, \gamma, \gamma')]_{\gamma, \gamma'}$ for any unramified character $\overline{\chi}$ (not necessarily exceptional) and simple reflection $\mathbf{w}_\alpha, \alpha \in \Delta$. The matrix is the analogue of Shahidi's local coefficient in the linear algebraic case, see [Sha, Chapter 5].

For Kazhdan-Patterson coverings $\overline{\text{GL}}_r^{(n)}$, the matrix $[\tau(\overline{\chi}, \mathbf{w}_\alpha, \gamma, \gamma')]_{\gamma, \gamma'}$ was firstly studied in [KP]. It also appeared in the work of Suzuki [Suz1], Chinta and Offen [CO] among others. For a subclass of Brylinski-Deligne covering groups, the study of matrix $[\tau(\overline{\chi}, \mathbf{w}_\alpha, \gamma, \gamma')]_{\gamma, \gamma'}$ was conducted in [Mc2] for unramified characters χ , generalizing that of Kazhdan and Patterson. Meanwhile, for ramified characters, it is included in the work of [GS]. However, in order to

work with the full class of Brylinski-Deligne covering groups and also remove the assumption $\mu_{2n} \subseteq F^\times$, we refine the computation in [Mc2] slightly. This is achieved by invoking the structural facts of Brylinski-Deligne covering groups, in particular those from §2.1. We also note that interesting phenomena dissipate when the assumption $\mu_{2n} \subseteq F^\times$ is imposed, for example for the type A_r case in §4. There are subtleties arising from the fact that -1 is not a square root. For this purpose, it is important to rigidify the formula for the matrix and express its entries in terms of naturally defined elements of the group.

Consider the Haar measure μ of F such that $\mu(O_F) = 1$. Thus, $\mu(O_F^\times) = 1 - 1/q$. Fix a chosen uniformizer ϖ of F . The Gauss sum is given by

$$G_\psi(a, b) = \int_{O_F^\times} (u, \varpi)_n^a \cdot \psi(\varpi^b u) \mu(u), \quad a, b \in \mathbf{Z}.$$

It is known that

$$G_\psi(a, b) = \begin{cases} 0 & \text{if } b < -1, \\ 1 - 1/q & \text{if } n|a, b \geq 0, \\ 0 & \text{if } n \nmid a, b \geq 0, \\ -1/q & \text{if } n|a, b = -1, \\ G_\psi(a, -1) \text{ with } |G_\psi(a, -1)| = q^{-1/2} & \text{if } n \nmid a, b = -1. \end{cases}$$

Recall $\varepsilon := \iota((-1, \varpi)_n) \in \mathbf{C}^\times$. One has $\overline{G_\psi(a, b)} = \varepsilon^a \cdot G_\psi(-a, b)$. For any $k \in \mathbf{Z}$, we write

$$\mathbf{g}_\psi(k) := G_\psi(k, -1).$$

As in [Mc2, §9], let $f_{\gamma'} \in I(\overline{\chi})$ be the function with $\text{supp}(f_{\gamma'}) = \overline{B}w_0K_1$, $f_{\gamma'}(w_0^{-1}) = i(\overline{\chi})(\gamma')\phi_0$ for a certain compact open subgroup K_1 . Here $\phi_0 \in i(\overline{\chi})^{T \cap K}$ is the unramified vector in $i(\overline{\chi})$. Recall from [Mc2, Corollary 9.2], one has $\tau(\overline{\chi}, \mathbf{w}_\alpha, \gamma, \gamma') = \langle \lambda_\gamma^{\mathbf{w}_\alpha \overline{\chi}}, T_{\mathbf{w}_\alpha, \overline{\chi}}(f_{\gamma'}) \rangle / |U^- \cap K_1|$. More precisely, from Equality (9.3) of [Mc2] one could evaluate $\tau(\overline{\chi}, \mathbf{w}_\alpha, \gamma, \gamma')$ by applying $\lambda_\gamma^{\mathbf{w}_\alpha \overline{\chi}} \in i(\mathbf{w}_\alpha \overline{\chi})^\vee$ to the integral

$$(6) \quad \int_F f_{\gamma'}(\overline{h}_\alpha(x^{-1}) \cdot \overline{e}_\alpha(-x) \cdot w_0^{-1}) \cdot \psi^{-1}(\overline{e}_\alpha(x^{-1})) \mu(x) \in i(\mathbf{w}_\alpha \overline{\chi}).$$

Note, the integrand of (6) takes values in $i(\overline{\chi})$. However, on one hand, as vector spaces of functions on \overline{T} , the underlying space $i(\overline{\chi})$ is identical to that of ${}^{w_\alpha}i(\overline{\chi})$ (cf. [Gao]); on the other hand, it follows from the Stone von-Neumann theorem that ${}^{w_\alpha}i(\overline{\chi}) \simeq i(\mathbf{w}_\alpha \overline{\chi})$ as representations of \overline{T} . Therefore, there is a canonical vector space isomorphism $i(\overline{\chi}) \simeq i(\mathbf{w}_\alpha \overline{\chi})$. For the computation below, we will follow [Mc2] closely and adopt this viewpoint implicitly.

To ease notations, write $\pi = i(\overline{\chi})$. Use the partition $F = \bigcup_{m \in \mathbf{Z}} \varpi^{-m} O_F^\times$ and write $x = \varpi^{-m} u^{-1}$, $u \in O_F^\times$. Then $\mu(x) = |\varpi|^{-m} \mu(u)$ and the integral in (6) is equal to

$$\begin{aligned} & \sum_{m \in \mathbf{Z}} |\varpi|^{-m} \int_{O_F^\times} f_{\gamma'}(\overline{h}_\alpha(\varpi^m \cdot u) \cdot \overline{e}_\alpha(-\varpi^{-m} u^{-1}) \cdot w_0^{-1}) \cdot \psi^{-1}(\overline{e}_\alpha(\varpi^m \cdot u)) \mu(u) \\ &= \sum_{m \in \mathbf{Z}} \int_{O_F^\times} (u, \varpi)_n^{mQ(\alpha^\vee)} \cdot \pi(\overline{h}_\alpha(\varpi^m)) \cdot \pi(\overline{h}_\alpha(u)) \cdot \pi(\gamma')\phi_0 \cdot \psi^{-1}(\varpi^m \cdot u) \mu(u) \end{aligned}$$

Suppose $\gamma' = \mathbf{s}_y \in \overline{T}$ for some $y \in Y$. (We write $\mathbf{s}_y := \mathbf{s}(\varpi^y) \in \overline{T}$ for $y \in Y$, see §2 for unconventional notations.) Then the above is equal to

$$(7) \quad \sum_{m \in \mathbf{Z}} \int_{O_F^\times} (u, \varpi)_n^{mQ(\alpha^\vee) + B(\alpha^\vee, y)} \cdot \pi(\overline{h}_\alpha(\varpi^m)) \cdot \pi(\mathbf{s}_y)\phi_0 \cdot \psi^{-1}(\varpi^m \cdot u) \mu(u)$$

From now, we write $\Gamma(m, y, \alpha^\vee) := \varepsilon^{(m + \langle y, \alpha \rangle)D(y, \alpha^\vee)}$ and $\mathbf{\Gamma}(y, \alpha^\vee) := \Gamma(-1, y, \alpha^\vee)$, which lie in $\{\pm 1\}$. It follows from the equality (3) that $\overline{h}_\alpha(\varpi^m) \cdot \mathbf{s}_y = w_\alpha \cdot (\Gamma(m, y, \alpha^\vee) \cdot \mathbf{s}_{y+m\alpha^\vee}) \cdot w_\alpha^{-1}$.

Therefore (7) is equal to

$$\sum_{m \in \mathbb{Z}} \Gamma(m, y, \alpha^\vee) \cdot {}^{w_\alpha} \pi \left(\mathbf{s}_{\mathbf{w}_\alpha(y+m\alpha^\vee)} \right) \phi_0 \cdot \int_{O_F^\times} (u, \varpi)_n^{mQ(\alpha^\vee)+B(\alpha^\vee, y)} \psi^{-1}(\varpi^m \cdot u) \mu(u).$$

There are three cases for each term in the sum:

- For $m \leq -2$, the integral over O_F^\times vanishes, and thus the contribution to $\tau(\overline{\chi}, \mathbf{w}_\alpha, \gamma, \gamma')$ is 0.

- For $m = -1$, the contribution $\tau(\overline{\chi}, \mathbf{w}_\alpha, \gamma, \gamma')$ is nonzero only when $\mathbf{w}_\alpha(y_1) \equiv y - \alpha^\vee \pmod{Y_{Q,n}}$ where $\gamma = \mathbf{s}_{y_1}, \gamma' = \mathbf{s}_y$. When $\mathbf{w}_\alpha(y_1) = y - \alpha^\vee$, the contribution to $\tau(\overline{\chi}, \mathbf{w}_\alpha, \gamma, \gamma')$ is

$$\mathbf{\Gamma}(y, \alpha^\vee) \cdot \mathbf{g}_{\psi^{-1}}(B(\alpha^\vee, y) - Q(\alpha^\vee)) = \mathbf{\Gamma}(y, \alpha^\vee) \cdot \mathbf{g}_{\psi^{-1}}(\langle y_\rho, \alpha \rangle Q(\alpha^\vee)).$$

- For any $x \in \mathbf{R}$, recall that we denote by $\lceil x \rceil$ the minimum integer such that $\lceil x \rceil \geq x$. The sum for $m \geq 0$ is equal to

$$\begin{aligned} & \sum_{m \geq 0} \Gamma(m, y, \alpha^\vee) \cdot {}^{w_\alpha} \pi \left(\mathbf{s}_{\mathbf{w}_\alpha(y+m\alpha^\vee)} \right) \phi_0 \cdot \int_{O_F^\times} (u, \varpi)_n^{mQ(\alpha^\vee)+B(\alpha^\vee, y)} \mu(u) \\ &= \sum_{\substack{m=k \cdot n_\alpha - B(\alpha^\vee, y)/Q(\alpha^\vee) \\ k \geq \lceil B(\alpha^\vee, y)/n_\alpha Q(\alpha^\vee) \rceil}} \Gamma(m, y, \alpha^\vee) \cdot \boldsymbol{\epsilon}^{(m+\langle y, \alpha \rangle)D(\alpha^\vee, y)} \\ & \quad \cdot {}^{w_\alpha} \pi \left(\mathbf{s}_{(-m-\langle y, \alpha \rangle)\alpha^\vee} \right) {}^{w_\alpha} \pi(\mathbf{s}_y) \phi_0 \cdot (1 - q^{-1}) \\ &= (1 - q^{-1}) \sum_{k \geq \lceil \langle y, \alpha^\vee \rangle / n_\alpha \rceil} \boldsymbol{\epsilon}^{kn_\alpha B(\alpha^\vee, y)} \cdot {}^{w_\alpha} \pi \left(\overline{h}_\alpha(\varpi^{-kn_\alpha}) \right) \cdot {}^{w_\alpha} \pi(\mathbf{s}_y) \phi_0 \\ &= (1 - q^{-1}) \sum_{k \geq \lceil \langle y, \alpha^\vee \rangle / n_\alpha \rceil} \overline{\chi}(\overline{h}_\alpha(\varpi^{n_\alpha}))^k \cdot {}^{w_\alpha} \pi(\mathbf{s}_y) \phi_0 \\ &= (1 - q^{-1}) \frac{\overline{\chi}(\overline{h}_\alpha(\varpi^{n_\alpha}))^{k_{y,\alpha}}}{1 - \overline{\chi}(\overline{h}_\alpha(\varpi^{n_\alpha}))} \cdot {}^{w_\alpha} \pi(\mathbf{s}_y) \phi_0, \text{ where } k_{y,\alpha} = \lceil \langle y, \alpha \rangle / n_\alpha \rceil. \end{aligned}$$

The contribution is nonzero only for $\gamma = \mathbf{s}_{y_1}$ with $y_1 \equiv y \pmod{Y_{Q,n}}$. In particular, if $y_1 = y$, then the contribution to $\tau(\overline{\chi}, \mathbf{w}_\alpha, \gamma, \gamma')$ (for $\gamma = \gamma' = \mathbf{s}_y$) is

$$(1 - q^{-1}) \frac{\overline{\chi}(\overline{h}_\alpha(\varpi^{n_\alpha}))^{k_{y,\alpha}}}{1 - \overline{\chi}(\overline{h}_\alpha(\varpi^{n_\alpha}))}, \text{ where } k_{y,\alpha} = \left\lceil \frac{\langle y, \alpha \rangle}{n_\alpha} \right\rceil.$$

To summarize, we state the following theorem by McNamara which generalizes [KP, Lemma I.3.3]:

Theorem 3.6 ([Mc2, Theorem 13.1]). *Suppose $\gamma = \mathbf{s}_{y_1}$ is represented by y_1 and $\gamma' = \mathbf{s}_y$ by y . Then we can write $\tau(\overline{\chi}, \mathbf{w}_\alpha, \gamma, \gamma') = \tau^1(\overline{\chi}, \mathbf{w}_\alpha, \gamma, \gamma') + \tau^2(\overline{\chi}, \mathbf{w}_\alpha, \gamma, \gamma')$ with the following properties:*

- $\tau^i(\overline{\chi}, \mathbf{w}_\alpha, \gamma \cdot \overline{z}, \gamma' \cdot \overline{z}') = (\mathbf{w}_\alpha \overline{\chi})^{-1}(\overline{z}) \cdot \tau^i(\overline{\chi}, \mathbf{w}_\alpha, \gamma, \gamma') \cdot \overline{\chi}(\overline{z}'), \quad \overline{z}, \overline{z}' \in \overline{A};$
- $\tau^1(\overline{\chi}, \mathbf{w}_\alpha, \gamma, \gamma') = 0$ unless $y_1 \equiv y \pmod{Y_{Q,n}};$
- $\tau^2(\overline{\chi}, \mathbf{w}_\alpha, \gamma, \gamma') = 0$ unless $y_1 \equiv \mathbf{w}_\alpha[y] \pmod{Y_{Q,n}}.$

Moreover,

- If $y_1 = y$, then

$$\tau^1(\overline{\chi}, \mathbf{w}_\alpha, \gamma, \gamma') = (1 - q^{-1}) \frac{\overline{\chi}(\overline{h}_\alpha(\varpi^{n_\alpha}))^{k_{y,\alpha}}}{1 - \overline{\chi}(\overline{h}_\alpha(\varpi^{n_\alpha}))}, \text{ where } k_{y,\alpha} = \left\lceil \frac{\langle y, \alpha \rangle}{n_\alpha} \right\rceil.$$

- If $y_1 = \mathbf{w}_\alpha[y]$, then

$$\tau^2(\overline{\chi}, \mathbf{w}_\alpha, \gamma, \gamma') = \mathbf{\Gamma}(y, \alpha^\vee) \cdot \mathbf{g}_{\psi^{-1}}(\langle y_\rho, \alpha \rangle Q(\alpha^\vee)).$$

Corollary 3.7. *Let $\overline{\chi}$ be an unramified exceptional character. Let $\lambda_{\mathbf{c}}^{\overline{\chi}} \in \text{Wh}_{\psi}(I(\overline{\chi}))$ be the Whittaker functional of $I(\overline{\chi})$ associated to some $\mathbf{c} \in \text{Ftn}(i(\overline{\chi}))$. Then, $\lambda_{\mathbf{c}}^{\overline{\chi}}$ lies in $\text{Wh}_{\psi}(\Theta(\overline{G}, \overline{\chi}))$ if and only if for all simple root $\alpha \in \Delta$,*

$$(9) \quad \mathbf{c}(\mathbf{s}_{\mathbf{w}_{\alpha}[y]}) = q^{k_{y,\alpha}-1} \cdot \Gamma(y, \alpha^{\vee}) \cdot \mathbf{g}_{\psi^{-1}}(\langle y_{\rho}, \alpha \rangle Q(\alpha^{\vee}))^{-1} \cdot \mathbf{c}(\mathbf{s}_y) \text{ for all } y.$$

Proof. By Corollary 3.5, for all $\alpha \in \Delta$, we have

$$\mathbf{c}(\mathbf{s}_y) \cdot \tau(\mathbf{w}_{\alpha}^{\overline{\chi}}, \mathbf{w}_{\alpha}, \mathbf{s}_y, \mathbf{s}_y) + \mathbf{c}(\mathbf{s}_{\mathbf{w}_{\alpha}[y]}) \cdot \tau(\mathbf{w}_{\alpha}^{\overline{\chi}}, \mathbf{w}_{\alpha}, \mathbf{s}_{\mathbf{w}_{\alpha}[y]}, \mathbf{s}_y) = 0,$$

where $y \in Y$ is any element. The preceding theorem gives

$$\begin{aligned} & \mathbf{c}(\mathbf{s}_{\mathbf{w}_{\alpha}[y]}) \\ &= - (1 - q^{-1}) \frac{(\overline{\chi}(\overline{h}_{\alpha}(\varpi^{n_{\alpha}})))^{-k_{y,\alpha}}}{1 - \overline{\chi}(\overline{h}_{\alpha}(\varpi^{n_{\alpha}}))^{-1}} \cdot \Gamma(y, \alpha^{\vee}) \cdot \mathbf{g}_{\psi^{-1}}(\langle y_{\rho}, \alpha \rangle Q(\alpha^{\vee}))^{-1} \cdot \mathbf{c}(\mathbf{s}_y) \\ &= q^{k_{y,\alpha}-1} \cdot \Gamma(y, \alpha^{\vee}) \cdot \mathbf{g}_{\psi^{-1}}(\langle y_{\rho}, \alpha \rangle Q(\alpha^{\vee}))^{-1} \cdot \mathbf{c}(\mathbf{s}_y). \end{aligned}$$

The proof is completed. \square

From now on, for $y \in Y$ and $\alpha \in \Delta$, we write

$$(9) \quad \mathbf{t}(\mathbf{w}_{\alpha}, y) := q^{k_{y,\alpha}-1} \cdot \Gamma(y, \alpha^{\vee}) \cdot \mathbf{g}_{\psi^{-1}}(\langle y_{\rho}, \alpha \rangle Q(\alpha^{\vee}))^{-1}$$

where

$$k_{y,\alpha} = \left\lceil \frac{\langle y_{\rho}, \alpha \rangle + 1}{n_{\alpha}} \right\rceil \text{ and } \Gamma(y, \alpha^{\vee}) = \varepsilon^{\langle y_{\rho}, \alpha \rangle \cdot D(y, \alpha^{\vee})}.$$

It is clear $\mathbf{t}(\mathbf{w}_{\alpha}, y) \neq 0$.

Definition 3.8. For $\mathbf{c} \in \text{Ftn}(i(\overline{\chi}))$, we say that \mathbf{c} vanishes on $y \in Y$ if and only if $\mathbf{c}(\mathbf{s}_y) = 0$. It is said to vanish on the orbit $\mathcal{O}_{y_0} \subset Y$ if and only if it vanishes on all $y \in \mathcal{O}_{y_0}$, in which case we write $\mathbf{c}(\mathcal{O}_{y_0}) = 0$.

Assume that \mathbf{c} gives rise to $\lambda_{\mathbf{c}}^{\overline{\chi}} \in \text{Wh}_{\psi}(\Theta(\overline{G}, \overline{\chi}))$. Since $\mathbf{t}(\mathbf{w}_{\alpha}, y) \neq 0$ for all y and $\alpha \in \Delta$, it follows from Corollary 3.7 that \mathbf{c} vanishes on \mathcal{O}_{y_0} if and only if it vanishes on any $y \in \mathcal{O}_{y_0}$. It is therefore easy to see

$$(10) \quad \dim \text{Wh}_{\psi}(\Theta(\overline{G}, \overline{\chi})) = \left| \left\{ \begin{array}{l} \mathcal{P}_{Q,n}(\mathcal{O}_{y_0}) : \\ \bullet \mathcal{O}_{y_0} \in \mathcal{O} \text{ is a } W\text{-orbit in } Y, \\ \bullet \text{ there exists } \mathbf{c} \in \text{Ftn}(i(\overline{\chi})) \\ \text{satisfying (8) for all } \alpha \in \Delta, y \in \mathcal{O}_{y_0} \\ \text{and also } \mathbf{c}(\mathcal{O}_{y_0}) \neq 0 \end{array} \right\} \right|.$$

In the remaining part of this section we will prove an effective lower and upper bound for $\dim \text{Wh}_{\psi}(\Theta(\overline{G}, \overline{\chi}))$.

3.4. A lower bound for $\dim \text{Wh}_{\psi}(\Theta(\overline{G}, \overline{\chi}))$. The Weyl group W of \mathbb{G} has the presentation

$$W = \langle \mathbf{w}_{\alpha} : (\mathbf{w}_{\alpha} \mathbf{w}_{\beta})^{m_{\alpha\beta}} = 1 \text{ for } \alpha, \beta \in \Delta \rangle.$$

Lemma 3.9. *Let $\mathcal{O}_y \in \mathcal{O}_{Q,n,sc}^F$ be a $Y_{Q,n}^{sc}$ -free orbit in Y . Then the following holds:*

$$\mathbf{t}(\mathbf{w}_{\alpha}, \mathbf{w}_{\alpha}[y]) \cdot \mathbf{t}(\mathbf{w}_{\alpha}, y) = 1 \text{ for all } \alpha \in \Delta.$$

Proof. Note $\mathbf{w}_{\alpha}[y] = \mathbf{w}_{\alpha}(y) + \alpha^{\vee} = y + (1 - \langle y, \alpha \rangle) \alpha^{\vee}$. It follows $\langle \mathbf{w}_{\alpha}[y], \alpha \rangle = 2 - \langle y, \alpha \rangle$. Therefore

$$\begin{aligned} & \mathbf{t}(\mathbf{w}_{\alpha}, \mathbf{w}_{\alpha}[y]) \\ &= q^{\lceil \langle \mathbf{w}_{\alpha}[y], \alpha \rangle / n_{\alpha} \rceil - 1} \cdot \Gamma(\mathbf{w}_{\alpha}[y], \alpha^{\vee}) \cdot \mathbf{g}_{\psi^{-1}}(Q(\alpha^{\vee})(\langle \mathbf{w}_{\alpha}[y], \alpha \rangle - 1))^{-1} \\ &= q^{\lceil (2 - \langle y, \alpha \rangle) / n_{\alpha} \rceil - 1} \cdot \varepsilon^{\langle y_{\rho}, \alpha \rangle \cdot (D(y, \alpha^{\vee}) - \langle y_{\rho}, \alpha^{\vee} \rangle Q(\alpha^{\vee}))} \cdot \mathbf{g}_{\psi^{-1}}(-Q(\alpha^{\vee}) \langle y_{\rho}, \alpha \rangle)^{-1} \end{aligned}$$

and

$$\begin{aligned} & \mathbf{t}(\mathbf{w}_\alpha, \mathbf{w}_\alpha[y]) \cdot \mathbf{t}(\mathbf{w}_\alpha, y) \\ &= q^{\lceil (2-\langle y, \alpha \rangle)/n_\alpha \rceil + \lceil \langle y, \alpha \rangle/n_\alpha \rceil - 2} \cdot \varepsilon^{\langle y_\rho, \alpha \rangle^2 \cdot Q(\alpha^\vee)} \\ & \quad \cdot \mathbf{g}_{\psi^{-1}}(Q(\alpha^\vee) \cdot \langle y_\rho, \alpha \rangle)^{-1} \cdot \mathbf{g}_{\psi^{-1}}(-Q(\alpha^\vee) \cdot \langle y_\rho, \alpha \rangle)^{-1}. \end{aligned}$$

However, it follows from $\mathbf{g}_{\psi^{-1}}(k) = \varepsilon^k \cdot \overline{\mathbf{g}_{\psi^{-1}}(-k)}$ that $|\mathbf{g}_{\psi^{-1}}(k)| = q^{-1/2}$. Moreover, since \mathcal{O}_y is a $Y_{Q,n}^{\text{sc}}$ -free orbit, $\mathbf{w}_\alpha[y] - y \notin Y_{Q,n}^{\text{sc}}$. Therefore, $n_\alpha \nmid (1 - \langle y, \alpha \rangle)$ and thus

$$\left\lceil \frac{2 - \langle y, \alpha \rangle}{n_\alpha} \right\rceil + \left\lceil \frac{\langle y, \alpha \rangle}{n_\alpha} \right\rceil = 1.$$

Now it can be checked easily that $\mathbf{t}(\mathbf{w}_\alpha, \mathbf{w}_\alpha[y]) \cdot \mathbf{t}(\mathbf{w}_\alpha, y) = 1$. \square

Now, consider adjacent $\alpha, \beta \in \Delta$ from the Dynkin diagram. We would like to show that for $Y_{Q,n}^{\text{sc}}$ -free orbit \mathcal{O}_y the equality $\prod_{i=1}^{m_{\alpha\beta}} \mathbf{t}(\mathbf{w}_\alpha \mathbf{w}_\beta, (\mathbf{w}_\alpha \mathbf{w}_\beta)^i[y]) = 1$ holds, where $\mathbf{t}(\mathbf{w}_\alpha \mathbf{w}_\beta, y) := \mathbf{t}(\mathbf{w}_\alpha, \mathbf{w}_\beta[y]) \cdot \mathbf{t}(\mathbf{w}_\beta, y)$. This will follow from a case by case discussion. We will give the details for $m_{\alpha\beta} = 3, 4$ below and leave the case for $m_{\alpha\beta} = 6$ to the reader.

CASE $m_{\alpha\beta} = 3$. The relation $(\mathbf{w}_\alpha \mathbf{w}_\beta)^{m_{\alpha\beta}} = 1$ is equivalent to $\mathbf{w}_\alpha \mathbf{w}_\beta \mathbf{w}_\alpha = \mathbf{w}_\beta \mathbf{w}_\alpha \mathbf{w}_\beta$. By Lemma 3.9, it suffices to show

$$(11) \quad \mathbf{t}(\mathbf{w}_\alpha, \mathbf{w}_\beta \mathbf{w}_\alpha[y]) \cdot \mathbf{t}(\mathbf{w}_\beta, \mathbf{w}_\alpha[y]) \cdot \mathbf{t}(\mathbf{w}_\alpha, y) = \mathbf{t}(\mathbf{w}_\beta, \mathbf{w}_\alpha \mathbf{w}_\beta[y]) \cdot \mathbf{t}(\mathbf{w}_\alpha, \mathbf{w}_\beta[y]) \cdot \mathbf{t}(\mathbf{w}_\beta, y).$$

Note

$$\mathbf{t}(\mathbf{w}_\alpha, y) = q^{\left\lceil \frac{\langle y_\rho, \alpha \rangle + 1}{n_\alpha} \right\rceil - 1} \cdot \varepsilon^{\langle y_\rho, \alpha \rangle \cdot D(y, \alpha^\vee)} \cdot \mathbf{g}_{\psi^{-1}}(B_Q(y_\rho, \alpha^\vee))^{-1}.$$

We also have $\langle \mathbf{w}_\beta \mathbf{w}_\alpha(y_\rho), \alpha \rangle = \langle y_\rho, \beta \rangle$ since the pairing $\langle -, - \rangle$ is W -equivariant and $\mathbf{w}_\alpha \mathbf{w}_\beta(\alpha) = \beta$. Similarly, $\langle \mathbf{w}_\alpha \mathbf{w}_\beta(y_\rho), \beta \rangle = \langle y_\rho, \alpha \rangle$. Simple computation gives

$$\begin{cases} \mathbf{t}(\mathbf{w}_\alpha, y) = q^{\left\lceil \frac{\langle y_\rho, \alpha \rangle + 1}{n_\alpha} \right\rceil - 1} \cdot \varepsilon^{\langle y_\rho, \alpha \rangle D(y, \alpha^\vee)} \cdot \mathbf{g}_{\psi^{-1}}(\langle y_\rho, \alpha \rangle Q(\alpha^\vee))^{-1}, \\ \mathbf{t}(\mathbf{w}_\beta, \mathbf{w}_\alpha[y]) = q^{\left\lceil \frac{\langle y_\rho, \alpha + \beta \rangle + 1}{n_\beta} \right\rceil - 1} \cdot \varepsilon^{\langle y_\rho, \alpha + \beta \rangle D(\mathbf{w}_\alpha[y], \beta^\vee)} \cdot \mathbf{g}_{\psi^{-1}}(\langle y_\rho, \alpha + \beta \rangle Q(\beta^\vee))^{-1}, \\ \mathbf{t}(\mathbf{w}_\alpha, \mathbf{w}_\beta \mathbf{w}_\alpha[y]) = q^{\left\lceil \frac{\langle y_\rho, \beta \rangle + 1}{n_\alpha} \right\rceil - 1} \cdot \varepsilon^{\langle y_\rho, \beta \rangle D(\mathbf{w}_\beta \mathbf{w}_\alpha[y], \alpha^\vee)} \cdot \mathbf{g}_{\psi^{-1}}(\langle y_\rho, \beta \rangle Q(\alpha^\vee))^{-1}. \end{cases}$$

Meanwhile, we also have

$$\begin{cases} \mathbf{t}(\mathbf{w}_\beta, y) = q^{\left\lceil \frac{\langle y_\rho, \beta \rangle + 1}{n_\beta} \right\rceil - 1} \cdot \varepsilon^{\langle y_\rho, \beta \rangle D(y, \beta^\vee)} \cdot \mathbf{g}_{\psi^{-1}}(\langle y_\rho, \beta \rangle Q(\beta^\vee))^{-1}, \\ \mathbf{t}(\mathbf{w}_\alpha, \mathbf{w}_\beta[y]) = q^{\left\lceil \frac{\langle y_\rho, \alpha + \beta \rangle + 1}{n_\alpha} \right\rceil - 1} \cdot \varepsilon^{\langle y_\rho, \alpha + \beta \rangle D(\mathbf{w}_\beta[y], \alpha^\vee)} \cdot \mathbf{g}_{\psi^{-1}}(\langle y_\rho, \alpha + \beta \rangle Q(\alpha^\vee))^{-1}, \\ \mathbf{t}(\mathbf{w}_\beta, \mathbf{w}_\alpha \mathbf{w}_\beta \mathbf{w}_\alpha[y]) = q^{\left\lceil \frac{\langle y_\rho, \alpha \rangle + 1}{n_\beta} \right\rceil - 1} \cdot \varepsilon^{\langle y_\rho, \alpha \rangle D(\mathbf{w}_\alpha \mathbf{w}_\beta[y], \beta^\vee)} \cdot \mathbf{g}_{\psi^{-1}}(\langle y_\rho, \alpha \rangle Q(\beta^\vee))^{-1}. \end{cases}$$

Note $Q(\alpha^\vee) = Q(\beta^\vee)$ and $n_\alpha = n_\beta$. To show the equality (11), it suffices to check that the powers of ε on the two sides of (11) are equal. However, straightforward computation shows that this is indeed the case, and we may omit the details.

CASE $m_{\alpha\beta} = 4$. Let $\alpha, \beta \in \Delta$ be two adjacent roots such that $m_{\alpha\beta} = 4$. We assume α is the longer one. Therefore $\langle \alpha^\vee, \beta \rangle = -1, \langle \beta^\vee, \alpha \rangle = -2$, and also $Q(\beta^\vee) = 2Q(\alpha^\vee)$. As in the preceding case, we want to show

$$(12) \quad \begin{aligned} & \mathbf{t}(\mathbf{w}_\beta, \mathbf{w}_\alpha \mathbf{w}_\beta \mathbf{w}_\alpha[y]) \cdot \mathbf{t}(\mathbf{w}_\alpha, \mathbf{w}_\beta \mathbf{w}_\alpha[y]) \cdot \mathbf{t}(\mathbf{w}_\beta, \mathbf{w}_\alpha[y]) \cdot \mathbf{t}(\mathbf{w}_\alpha, y) \\ &= \mathbf{t}(\mathbf{w}_\alpha, \mathbf{w}_\beta \mathbf{w}_\alpha \mathbf{w}_\beta[y]) \cdot \mathbf{t}(\mathbf{w}_\beta, \mathbf{w}_\alpha \mathbf{w}_\beta[y]) \cdot \mathbf{t}(\mathbf{w}_\alpha, \mathbf{w}_\beta[y]) \cdot \mathbf{t}(\mathbf{w}_\beta, y). \end{aligned}$$

Simple computation yields

$$\begin{cases} \mathbf{t}(\mathbf{w}_\alpha, y) = q^{\left\lfloor \frac{\langle y_\rho, \alpha \rangle + 1}{n_\alpha} \right\rfloor - 1} \cdot \boldsymbol{\varepsilon}^{\langle y_\rho, \alpha \rangle D(y, \alpha^\vee)} \cdot \mathbf{g}_{\psi^{-1}}(\langle y_\rho, \alpha \rangle Q(\alpha^\vee))^{-1}, \\ \mathbf{t}(\mathbf{w}_\beta, \mathbf{w}_\alpha[y]) = q^{\left\lfloor \frac{\langle y_\rho, \alpha + \beta \rangle + 1}{n_\beta} \right\rfloor - 1} \cdot \boldsymbol{\varepsilon}^{\langle y_\rho, \alpha + \beta \rangle D(\mathbf{w}_\alpha[y], \beta^\vee)} \cdot \mathbf{g}_{\psi^{-1}}(\langle y_\rho, \alpha + \beta \rangle Q(\beta^\vee))^{-1}, \\ \mathbf{t}(\mathbf{w}_\alpha, \mathbf{w}_\beta \mathbf{w}_\alpha[y]) = q^{\left\lfloor \frac{\langle y_\rho, \alpha + 2\beta \rangle + 1}{n_\alpha} \right\rfloor - 1} \cdot \boldsymbol{\varepsilon}^{\langle y_\rho, \alpha + 2\beta \rangle D(\mathbf{w}_\beta \mathbf{w}_\alpha[y], \alpha^\vee)} \cdot \mathbf{g}_{\psi^{-1}}(\langle y_\rho, \alpha + 2\beta \rangle Q(\alpha^\vee))^{-1}, \\ \mathbf{t}(\mathbf{w}_\beta, \mathbf{w}_\alpha \mathbf{w}_\beta \mathbf{w}_\alpha[y]) = q^{\left\lfloor \frac{\langle y_\rho, \beta \rangle + 1}{n_\beta} \right\rfloor - 1} \cdot \boldsymbol{\varepsilon}^{\langle y_\rho, \beta \rangle D(\mathbf{w}_\alpha \mathbf{w}_\beta \mathbf{w}_\alpha[y], \beta^\vee)} \cdot \mathbf{g}_{\psi^{-1}}(\langle y_\rho, \beta \rangle Q(\beta^\vee))^{-1}. \end{cases}$$

On the other hand, for the right hand side of (12), one has

$$\begin{cases} \mathbf{t}(\mathbf{w}_\beta, y) = q^{\left\lfloor \frac{\langle y_\rho, \beta \rangle + 1}{n_\beta} \right\rfloor - 1} \cdot \boldsymbol{\varepsilon}^{\langle y_\rho, \beta \rangle D(y, \beta^\vee)} \cdot \mathbf{g}_{\psi^{-1}}(\langle y_\rho, \beta \rangle Q(\beta^\vee))^{-1}, \\ \mathbf{t}(\mathbf{w}_\alpha, \mathbf{w}_\beta[y]) = q^{\left\lfloor \frac{\langle y_\rho, \alpha + 2\beta \rangle + 1}{n_\alpha} \right\rfloor - 1} \cdot \boldsymbol{\varepsilon}^{\langle y_\rho, \alpha + 2\beta \rangle D(\mathbf{w}_\beta[y], \alpha^\vee)} \cdot \mathbf{g}_{\psi^{-1}}(\langle y_\rho, \alpha + 2\beta \rangle Q(\alpha^\vee))^{-1}, \\ \mathbf{t}(\mathbf{w}_\beta, \mathbf{w}_\alpha \mathbf{w}_\beta \mathbf{w}_\alpha[y]) = q^{\left\lfloor \frac{\langle y_\rho, \alpha + \beta \rangle + 1}{n_\beta} \right\rfloor - 1} \cdot \boldsymbol{\varepsilon}^{\langle y_\rho, \alpha + \beta \rangle D(\mathbf{w}_\alpha \mathbf{w}_\beta[y], \beta^\vee)} \cdot \mathbf{g}_{\psi^{-1}}(\langle y_\rho, \alpha + \beta \rangle Q(\beta^\vee))^{-1}, \\ \mathbf{t}(\mathbf{w}_\alpha, \mathbf{w}_\beta \mathbf{w}_\alpha \mathbf{w}_\beta[y]) = q^{\left\lfloor \frac{\langle y_\rho, \alpha \rangle + 1}{n_\alpha} \right\rfloor - 1} \cdot \boldsymbol{\varepsilon}^{\langle y_\rho, \alpha \rangle D(\mathbf{w}_\beta \mathbf{w}_\alpha \mathbf{w}_\beta[y], \alpha^\vee)} \cdot \mathbf{g}_{\psi^{-1}}(\langle y_\rho, \alpha \rangle Q(\alpha^\vee))^{-1}. \end{cases}$$

To show the equality (12), again it suffices to show the powers of $\boldsymbol{\varepsilon}$ of the two sides have the same parities, and this is achieved from a straightforward check.

Analogous argument for $m_{\alpha\beta} = 6$ works, and we have:

Proposition 3.10. *Let \mathcal{O}_y be a $Y_{Q,n}^{\text{sc}}$ -free orbit. For all adjacent $\alpha, \beta \in \Delta$, one has*

$$\prod_{i=1}^{m_{\alpha\beta}} \mathbf{t}(\mathbf{w}_\alpha \mathbf{w}_\beta, (\mathbf{w}_\alpha \mathbf{w}_\beta)^i[y]) = 1,$$

where $\mathbf{t}(\mathbf{w}_\alpha \mathbf{w}_\beta, y) := \mathbf{t}(\mathbf{w}_\alpha, \mathbf{w}_\beta[y]) \cdot \mathbf{t}(\mathbf{w}_\beta, y)$.

Definition 3.11. Let $\mathcal{O}_y \in \mathcal{O}_{Q,n}^F$ be a $Y_{Q,n}^{\text{sc}}$ -free orbit. For any $\mathbf{w} = \mathbf{w}_k \mathbf{w}_{k-1} \dots \mathbf{w}_2 \mathbf{w}_1 \in W$ written as a minimum expansion, write $\mathbf{T}(\mathbf{w}, y) := \prod_{i=1}^k \mathbf{t}(\mathbf{w}_i, \mathbf{w}_{i-1} \dots \mathbf{w}_1[y])$, which, by Lemma 3.9 and Proposition 3.10, is independent of the choice of minimum expansion of \mathbf{w} .

Let $\mathcal{O}_y \in \mathcal{O}_{Q,n}^F$ be a $Y_{Q,n}$ -free orbit (and therefore $Y_{Q,n}^{\text{sc}}$ -free). We define a nonzero \mathbf{c} with support \mathcal{O}_y as follows. First, let $\mathbf{c}(\mathbf{s}_y) = 1$, and for any $\alpha \in \Delta$, let

$$\mathbf{c}(\mathbf{s}_{\mathbf{w}_\alpha[y]}) := \mathbf{t}(\mathbf{w}_\alpha, y) \cdot \mathbf{c}(\mathbf{s}_y).$$

Inductively, one can define $\mathbf{c}(\mathbf{s}_{\mathbf{w}[y]}) := \mathbf{T}(\mathbf{w}, y) \cdot \mathbf{c}(\mathbf{s}_y)$ for any $\mathbf{w} \in W$. It is well-defined and independent of the minimum decomposition of \mathbf{w} . Second, extend \mathbf{c} by

$$\mathbf{c}(\mathbf{s}_{\mathbf{w}[y]} \cdot \bar{\mathbf{z}}) = \mathbf{c}(\mathbf{s}_{\mathbf{w}[y]}) \cdot \bar{\chi}(\bar{\mathbf{z}}), \quad \bar{\mathbf{z}} \in \bar{A}.$$

and

$$\mathbf{c}(\bar{t}) = 0 \text{ if } \bar{t} \notin \bigcup_{\mathbf{w} \in W} \mathbf{s}_{\mathbf{w}[y]} \cdot \bar{A}.$$

By using the property that $\mathbf{T}(\mathbf{w}, y)$ and $\mathbf{c}(\mathbf{s}_{\mathbf{w}[y]})$ are independent of the minimum decomposition of \mathbf{w} , we see that the equality (8) is satisfied. It follows that $\wp_{Q,n}(\mathcal{O}_y)$ belongs to the right hand side of the equality (10). Therefore,

$$(13) \quad \dim \text{Wh}_\psi(\Theta(\bar{G}, \bar{\chi})) \geq \left| \wp_{Q,n}(\mathcal{O}_{Q,n}^F) \right|.$$

3.5. An upper bound for $\dim \text{Wh}_\psi(\Theta(\overline{G}, \overline{\chi}))$. First we show a result in the general setting regarding the usual Weyl action. Let Ψ be a root system and Ψ_s be a fixed choice of simple roots. Write $L := \langle \Psi \rangle$ for the lattice generated by Ψ and $V = L \otimes \mathbf{R}$. The Weyl group W associated to Ψ acts on V naturally by the usual linear transformation generated by simple reflections. Recall, we write $\mathfrak{w}(v)$, $\mathfrak{w} \in W, v \in V$ for this action.

Lemma 3.12. *Let $v \in V$ be any vector such that $\mathfrak{w}(v) \equiv v \pmod{L}$. Then there exists $\mathfrak{w}' \in W$ and $\alpha \in \Psi_s$ such that $\mathfrak{w}_\alpha(\mathfrak{w}'(v)) \equiv \mathfrak{w}'(v) \pmod{L}$.*

Proof. Let $W_{\text{aff}} = L \rtimes W$ be the affine Weyl group, and denote any element of W_{aff} by $\mathfrak{w}_a = (y, \mathfrak{w})$. We call \mathfrak{w} the Weyl component of \mathfrak{w}_a . The congruence $\mathfrak{w}(v) \equiv v \pmod{L}$ is equivalent to $\mathfrak{w}_a(v) = v$ for some \mathfrak{w}_a which projects to $\mathfrak{w} \in W$.

If $\mathfrak{w}_a(v) = v$, it then follows that $v \in V$ lies on the boundary of \overline{C} , where C is an alcove (i.e. a fundamental domain) of the action of W_{aff} on V , see [Bou]. Note \overline{C} is a simplicial complex whose boundary consists of $|\Psi_s| + 1$ walls $\{E_i\}$. Moreover, we may assume that for $1 \leq i \leq |\Psi_s|$, the wall E_i lies in the hyperplane fixed by \mathfrak{w}_a whose Weyl component is \mathfrak{w}_{α_i} for some $\alpha_i \in \Psi_s$. In this case, one also knows that $E_{|\Psi_s|+1}$ is fixed by $(y, \mathfrak{w}_\beta) \in W_{\text{aff}}$ for some $\beta \in \Psi - \Psi_s$.

Since $v \in \bigcup_i E_i$, there are two cases. First, suppose $v \in E_i$ for some $1 \leq i \leq |\Psi_s|$; then clearly $\mathfrak{w}_{\alpha_i}(v) \equiv v \pmod{L}$ for some $\alpha_i \in \Psi_s$. Otherwise, suppose $v \in E_{|\Psi_s|+1}$. Let $\mathfrak{w}' \in W$ be such that $\mathfrak{w}'(\beta) \in \Psi_s$. It follows that $\mathfrak{w}'(E_{|\Psi_s|+1})$ is fixed by some $\mathfrak{w}_a = (y, \mathfrak{w}_\alpha)$ with $\alpha \in \Psi_s$. That is, $\mathfrak{w}_\alpha(\mathfrak{w}'(v)) \equiv \mathfrak{w}'(v) \pmod{L}$. The proof is completed. \square

Proposition 3.13. *Consider $\mathbf{c} \in \text{Ftn}(i(\overline{\chi}))$ such that $\lambda_{\mathbf{c}}^{\overline{\chi}}$ is a Whittaker functional on $\Theta(\overline{G}, \overline{\chi})$. If \mathcal{O}_{y^0} is not $Y_{Q,n}^{\text{sc}}$ -free, then \mathbf{c} is zero on \mathcal{O}_{y^0} . It follows $\dim \text{Wh}_\psi(\Theta(\overline{G}, \overline{\chi})) \leq \left| \wp_{Q,n}(\mathcal{O}_{Q,n,\text{sc}}^F) \right|$.*

Proof. Write $V = Y \otimes \mathbf{R}$. One has $V = (Y^{\text{sc}} \otimes \mathbf{R}) \oplus V_0$ where $V_0 \subseteq V$ is fixed by W pointwise with respect to the usual action, i.e., the action $\mathfrak{w}(v)$ of W . In general $y_\rho^0 \in V$; however, without loss of generality, we may assume $y_\rho^0 \in Y^{\text{sc}} \otimes \mathbf{R}$ now. There is a canonical W -equivariant isomorphism $Y_{Q,n}^{\text{sc}} \otimes \mathbf{R} \simeq Y^{\text{sc}} \otimes \mathbf{R}$ with respect to the usual action. Moreover, $\left\{ \alpha_{Q,n}^\vee \right\}_{\alpha \in \Phi}$ forms a root system.

If \mathcal{O}_{y^0} is not $Y_{Q,n}^{\text{sc}}$ -free, then there exists $\mathfrak{w} \in W$ such that $\mathfrak{w}[y^0] \equiv y^0 \pmod{Y_{Q,n}^{\text{sc}}}$, i.e. $\mathfrak{w}(y_\rho^0) \equiv y_\rho^0 \pmod{Y_{Q,n}^{\text{sc}}}$. By the preceding Lemma, there exists $y \in \mathcal{O}_{y^0}$ and $\alpha \in \Delta$ such that $\mathfrak{w}_\alpha(y_\rho) \equiv y_\rho \pmod{Y_{Q,n}^{\text{sc}}}$. Now it suffices to show that \mathbf{c} vanishes on y .

By Corollary 3.7, $\mathbf{c}(\mathbf{s}_{\mathfrak{w}_\alpha[y]}) = \mathbf{t}(\mathfrak{w}_\alpha, y) \cdot \mathbf{c}(\mathbf{s}_y)$. Since $\mathfrak{w}_\alpha(y_\rho) \equiv y_\rho \pmod{Y_{Q,n}^{\text{sc}}}$, it follows $n_\alpha | \langle y_\rho, \alpha \rangle$. Write $\langle y_\rho, \alpha \rangle = k \cdot n_\alpha$. Since $\mathbf{s}_{\mathfrak{w}_\alpha[y]} = \mathbf{s}_y \cdot \mathbf{s}_{-\langle y_\rho, \alpha \rangle \alpha^\vee} \cdot \boldsymbol{\varepsilon}^{\langle y_\rho, \alpha \rangle \cdot D(\alpha^\vee, y)}$, one has

$$\begin{aligned} & \mathbf{c}(\mathbf{s}_{\mathfrak{w}_\alpha[y]}) \\ &= \overline{\chi}(\mathbf{s}_{-kn_\alpha \alpha^\vee}) \cdot \mathbf{c}(\mathbf{s}_y) \cdot \boldsymbol{\varepsilon}^{\langle y_\rho, \alpha \rangle \cdot D(\alpha^\vee, y)} \\ &= q^k \cdot \boldsymbol{\varepsilon}^{kn_\alpha \cdot D(\alpha^\vee, y)} \cdot \mathbf{c}(\mathbf{s}_y). \end{aligned}$$

On the other hand,

$$\begin{aligned} & \mathbf{t}(\mathfrak{w}_\alpha, y) \cdot \mathbf{c}(\mathbf{s}_y) \\ &= q^{k_{y, \alpha^{-1}}} \cdot \boldsymbol{\Gamma}(y, \alpha^\vee) \cdot \mathbf{g}_{\psi^{-1}}(\langle y_\rho, \alpha \rangle Q(\alpha^\vee))^{-1} \cdot \mathbf{c}(\mathbf{s}_y) \\ &= q^k \cdot (-1, \varpi)_n^{kn_\alpha \cdot D(y, \alpha^\vee)} \cdot (-q^{-1}) \cdot \mathbf{c}(\mathbf{s}_y). \end{aligned}$$

It follows $\mathbf{c}(\mathbf{s}_y) = -q^{-1} \cdot \boldsymbol{\varepsilon}^{kn_\alpha B(y, \alpha^\vee)} \cdot \mathbf{c}(\mathbf{s}_y) = (-q^{-1}) \cdot \mathbf{c}(\mathbf{s}_y)$. Therefore $\mathbf{c}(\mathbf{s}_y) = 0$. The proof is completed. \square

Theorem 3.14. *Let \overline{G} be an unramified Brylinski-Deligne covering group incarnated by (D, η) . Let $\overline{\chi}$ be an unramified exceptional character and $\Theta(\overline{G}, \overline{\chi})$ the theta representation associated with $\overline{\chi}$. Then*

$$\left| \wp_{Q,n}(\mathcal{O}_{Q,n}^F) \right| \leq \dim \text{Wh}_\psi(\Theta(\overline{G}, \overline{\chi})) \leq \left| \wp_{Q,n}(\mathcal{O}_{Q,n,\text{sc}}^F) \right|.$$

The center $Z(\overline{G}^\vee)$ of the dual group \overline{G}^\vee of \overline{G} is identified with $\text{Hom}(Y_{Q,n}/Y_{Q,n}^{\text{sc}}, \mathbf{C}^\times)$. Therefore $Y_{Q,n}^{\text{sc}} = Y_{Q,n}$ if and only if $Z(\overline{G}^\vee) = \{1\}$. Immediately it follows:

Corollary 3.15. *If the dual group \overline{G}^\vee of \overline{G} is of adjoint type, i.e. $Z(\overline{G}^\vee) = 1$, then $\dim \text{Wh}_\psi(\Theta(\overline{G}, \overline{\chi})) = |\wp_{Q,n}(\mathcal{O}_{Q,n}^F)|$.*

For groups of type E_8, F_4 and G_2 , the complex dual group of their covering group has trivial center and thus Corollary 3.15 applies.

More generally, if $\mathcal{O}_{Q,n}^F = \mathcal{O}_{Q,n,\text{sc}}^F$, then the dimension of $\text{Wh}_\psi(\Theta(\overline{G}, \overline{\chi}))$ can be uniquely determined. We will illustrate below that Theorem 3.14 recovers the result of Kazhdan-Patterson in this case.

Example 3.16. Let $\{e_1, e_2, \dots, e_r\}$ be a basis for the cocharacter lattice Y of GL_r . The simple coroots Δ^\vee of GL_r are $\Delta^\vee = \{\alpha_i^\vee := e_i - e_{i+1}\}_{1 \leq i \leq r-1}$. The isomorphism class of (D, η) in the incarnation category corresponds to a Weyl-invariant quadratic form Q , or equivalently, to the bilinear form B_Q . Let $B_Q(e_i, e_j)$ be the Weyl-invariant bilinear form determined by

$$B_Q(e_i, e_i) = 2\mathbf{p}, \quad B_Q(e_i, e_j) = \mathbf{q} \text{ if } i \neq j.$$

For any root α , one has $Q(\alpha^\vee) = 2\mathbf{p} - \mathbf{q}$. We assume $2\mathbf{p} - \mathbf{q} = -1$ and therefore $n_\alpha = n$. The covering groups $\overline{\text{GL}}_r^{(n)}$ arising from such B_Q are exactly those studied by Kazhdan-Patterson. The parameter \mathbf{p} corresponds to the twisting parameter c in [KP].

From B_Q , the lattice $Y_{Q,n}$ is given by

$$\left\{ \sum_i x_i e_i \in \bigoplus_{i=1}^r \mathbf{Z} e_i : x_1 \equiv x_2 \equiv \dots \equiv x_r \pmod{n}, \text{ and } n | (\mathbf{q}r - 1)x_i \right\}.$$

The lattice $Y_{Q,n}^{\text{sc}}$ is generated by $\{\alpha_{Q,n}^\vee\}_{\alpha \in \Phi}$. It is easy to check $Y_{Q,n}^{\text{sc}} = Y_{Q,n} \cap Y^{\text{sc}}$, and this has the following implications.

Suppose \mathcal{O}_y is a not $Y_{Q,n}$ -free, i.e. $\mathbf{w}[y] - y \in Y_{Q,n}$ for some $\mathbf{w} \neq 1 \in W$. Clearly $\mathbf{w}[y] - y \in Y^{\text{sc}}$ as well. It follows $\mathbf{w}[y] - y \in Y_{Q,n}^{\text{sc}}$, i.e., \mathcal{O}_y is not $Y_{Q,n}^{\text{sc}}$ -free. Therefore, for the Kazhdan-Patterson covering group $\overline{\text{GL}}_r^{(n)}$, one has $\mathcal{O}_{Q,n}^F = \mathcal{O}_{Q,n,\text{sc}}^F$. Consequently, for the covering group $\overline{\text{GL}}_r^{(n)}$ with parameter (\mathbf{p}, \mathbf{q}) such that $2\mathbf{p} - \mathbf{q} = -1$, Theorem 3.14 yields

$$\dim \text{Wh}_\psi(\Theta(\overline{G}, \overline{\chi})) = |\wp_{Q,n}(\mathcal{O}_{Q,n,\text{sc}}^F)|,$$

which is the content of [KP, Theorem I.3.5]. Moreover, distinguished theta representations (cf. Definition 3.3) for $\overline{\text{GL}}_r^{(n)}$ are completely determined in [KP, Corollary I.3.6].

In the remaining part of the paper, we will determine the distinguished theta representations for coverings of simply-connected groups of type A_r, B_r, C_r and G_2 . To ease the computations, we will use the standard coordinates for the coroot system of each type as in [Bou, pp. 265-290].

4. THE $A_r, r \geq 1$ CASE

Consider the Dynkin diagram for the simple coroots of A_r :

$$\alpha_1^\vee \text{ --- } \alpha_2^\vee \text{ --- } \dots \text{ --- } \alpha_{r-2}^\vee \text{ --- } \alpha_{r-1}^\vee \text{ --- } \alpha_r^\vee$$

The cocharacter lattice is $Y = Y^{\text{sc}} = \bigoplus_{i=1}^r \mathbf{Z} \alpha_i^\vee$. As in [Bou, page 265], consider the embedding $\mathbf{i}_A : \bigoplus_{i=1}^r \mathbf{Z} \alpha_i^\vee \rightarrow \bigoplus_{i=1}^{r+1} \mathbf{Z} e_i$, which is given by

$$\mathbf{i}_A : y = (x_1, x_2, \dots, x_r) \mapsto \mathbf{i}_A(y) = (x_1, x_2 - x_1, x_3 - x_2, \dots, x_r - x_{r-1}, -x_r).$$

In particular, we can identify the image of \mathbf{i}_A : any $(y_1, y_2, \dots, y_r, y_{r+1}) \in \bigoplus_{i=1}^{r+1} \mathbf{Z} e_i$ is equal to $\mathbf{i}_A(y)$ for some y if and only if $\sum_{i=1}^{r+1} y_i = 0$.

Meanwhile, $\rho = \sum_{i=1}^r \frac{i(r-i+1)}{2} \alpha_i^\vee$. We use $\mathbf{i}_A : \bigoplus_{i=1}^r \mathbf{Q} \alpha_i^\vee \rightarrow \bigoplus_{i=1}^{r+1} \mathbf{Q} e_i$ to denote the canonical extension of \mathbf{i}_A . Then, $\mathbf{i}_A(\rho) = (r/2, (r-2)/2, \dots, -(r-2)/2, -r/2) \in \bigoplus_{i=1}^{r+1} \mathbf{Q} e_i$.

It follows that for any $y \in Y$,

$$\begin{aligned} & \mathbf{i}_A(y_\rho) \\ &= (x_1 - \frac{r}{2}, \dots, x_i - x_{i-1} + (i-1) - \frac{r}{2}, \dots, -x_r + r - \frac{r}{2}), \quad 1 \leq i \leq r \\ &= (x_1, x_2 - x_1 + 1, \dots, x_i - x_{i-1} + (i-1), \dots, -x_r + r) + (-r/2, -r/2, \dots, -r/2). \end{aligned}$$

From now, we write $\mathbf{i}_A^*(y_\rho) := (x_1^*, x_2^*, \dots, x_r^*, x_{r+1}^*)$ for $(x_1, x_2 - x_1 + 1, \dots, x_i - x_{i-1} + (i-1), \dots, -x_r + r) \in \bigoplus_i \mathbf{Z} e_i$. Thus,

$$\mathbf{i}_A(y_\rho) = \mathbf{i}_A^*(y_\rho) + (-r/2, -r/2, \dots, -r/2).$$

Meanwhile, any $(x_1^*, x_2^*, \dots, x_r^*, x_{r+1}^*) \in \bigoplus_i \mathbf{Z} e_i$ is equal to $\mathbf{i}_A^*(y_\rho)$ for some y if and only if $\sum_{i=1}^{r+1} x_i^* = r(r+1)/2$.

Consider the quadratic form Q on $Y = \langle \alpha_i^\vee, 1 \leq i \leq r \rangle$ with $Q(\alpha_i^\vee) = 1$ for all i . Then

$$B_Q(\alpha_i^\vee, \alpha_j^\vee) = \begin{cases} 2, & \text{if } i = j; \\ -1, & \text{if } j = i + 1; \\ 0, & \text{if } \alpha_i^\vee, \alpha_j^\vee \text{ are not adjacent.} \end{cases}$$

This gives rise to the degree n covering group $\overline{\text{SL}}_{r+1}^{(n)}$. Any element $\sum_{i=1}^r x_i \alpha_i^\vee \in Y$ lies in $Y_{Q,n}$ if and only if

$$\begin{cases} 2x_1 - x_2 \\ -x_1 + 2x_2 - x_3 \\ -x_2 + 2x_3 - x_4 \\ \dots \\ -x_{r-2} + 2x_{r-1} - x_r \\ -x_{r-1} + 2x_r \end{cases} \in n\mathbf{Z}.$$

By using \mathbf{i}_A , we see

$$Y_{Q,n} = \left\{ \begin{array}{l} (y_1, y_2, \dots, y_r) \in \bigoplus_{i=1}^{r+1} \mathbf{Z} e_i : \\ \bullet \sum_{i=1}^{r+1} y_i = 0, \\ \bullet y_1 \equiv \dots \equiv y_r \equiv y_{r+1} \pmod{n} \end{array} \right\} \text{ and } Y_{Q,n}^{\text{sc}} = \left\{ \begin{array}{l} (y_1, y_2, \dots, y_r) \in \bigoplus_{i=1}^{r+1} \mathbf{Z} e_i : \\ \bullet \sum_{i=1}^{r+1} y_i = 0, \\ \bullet n|y_i \text{ for all } i. \end{array} \right\}$$

The Weyl group $W = S_{r+1}$ acts as permutations on $\bigoplus_{i=1}^{r+1} \mathbf{Z} e_i$. In particular, $w_{\alpha_i}, \alpha_i \in \Delta$ acts by exchanging e_i and e_{i+1} .

4.1. Case I: $\overline{\text{SL}}_{r+1}^{(n)}, n \leq r$. Suppose $n \leq r$, then for any $y \in Y$ with $\mathbf{i}_A^*(y_\rho) = (x_1^*, x_2^*, \dots, x_{r+1}^*)$, there exists $x_i^*, x_j^*, i \neq j$ such that $n|(x_i^* - x_j^*)$. Then clearly $w(y_\rho) - y_\rho \in Y_{Q,n}^{\text{sc}}$ for some $w \in W$. That is, $\mathcal{O}_y \notin \mathcal{O}_{Q,n,\text{sc}}^F$ and one has in this case

$$\mathcal{O}_{Q,n,\text{sc}}^F = \emptyset.$$

Therefore, $\dim \text{Wh}_\psi(\Theta(\overline{\text{SL}}_{r+1}^{(n)}, \overline{\chi})) = 0$ for $n \leq r$.

4.2. Case II: $\overline{\text{SL}}_{r+1}^{(n)}, n = r+1$. Note in this case, the dual group for $\overline{\text{SL}}_n^{(n)}$ is SL_n , see [We2]. Consider $\mathcal{O}_y \in \mathcal{O}_{Q,n,\text{sc}}^F$ such that $\mathbf{i}_A^*(y_\rho) = (0, 1, 2, \dots, r-1, r) \in \bigoplus_{i=1}^{r+1} \mathbf{Z} e_i$. It is easy to check $\wp_{Q,n}^{\text{sc}}(\mathcal{O}_{Q,n,\text{sc}}^F) = \{\wp_{Q,n}^{\text{sc}}(\mathcal{O}_y)\}$, and this implies $|\wp_{Q,n}(\mathcal{O}_{Q,n,\text{sc}}^F)| = 1$. However, $\mathcal{O}_y \notin \mathcal{O}_{Q,n}^F$. For example, let $w_{\mathfrak{h}}$ be such that $\mathbf{i}_A^*(w_{\mathfrak{h}}(y_\rho)) = (1, 2, \dots, r, 0)$, then $\mathbf{i}_A(w_{\mathfrak{h}}(y_\rho)) - \mathbf{i}_A(y_\rho) = (1, 1, \dots, 1, -r) \in Y_{Q,n}$. That is, $w_{\mathfrak{h}}[y] - y \in Y_{Q,n}$. Therefore,

$$|\wp_{Q,n}(\mathcal{O}_{Q,n}^F)| = 0.$$

It follows $0 \leq \dim \text{Wh}_\psi(\Theta(\overline{\text{SL}}_n^{(n)}, \bar{\chi})) \leq 1$, where $n = r + 1$. In this case, it is delicate to determine $\dim \text{Wh}_\psi(\Theta(\overline{\text{SL}}_n^{(n)}, \bar{\chi}))$, and there are additional constraints on the exceptional character $\bar{\chi}$ such that $\Theta(\overline{\text{SL}}_n^{(n)}, \bar{\chi})$ is distinguished. The analysis below is devoted to this.

4.2.1. *The reduction step.* It is clear that $\mathbf{i}_A^*(y_\rho) = (0, 1, 2, \dots, r-1, r)$ if and only if $y = 0$. Moreover, $\mathbf{i}_A^*(\mathbf{w}_\natural(y_\rho)) = (1, 2, 3, \dots, r, 0)$ for $\mathbf{w}_\natural = \mathbf{w}_{\alpha_r} \mathbf{w}_{\alpha_{r-1}} \dots \mathbf{w}_{\alpha_2} \mathbf{w}_{\alpha_1}$. As above,

$$\mathbf{w}_\natural[0] - 0 = \sum_{i=1}^r i \cdot \alpha_i^\vee \in Y_{Q,n}.$$

Write $y_{Q,n} := \sum_{i=1}^r i \cdot \alpha_i^\vee$. In fact, the set $\{n\alpha_i^\vee : 2 \leq i \leq r\} \cup \{y_{Q,n}\}$ forms a basis for $Y_{Q,n}$, whereas $\{n\alpha_i^\vee : 2 \leq i \leq r\} \cup \{n \cdot y_{Q,n}\}$ is a basis for $Y_{Q,n}^{\text{sc}}$. It follows that any exceptional character $\bar{\chi}$ is determined by its value at $\mathbf{s}_{y_{Q,n}}$.

We choose the bisector D on Y^{sc} such that $D(\alpha_i^\vee, \alpha_j^\vee)$ is given by

$$D(\alpha_i^\vee, \alpha_j^\vee) = \begin{cases} Q(\alpha_i^\vee) & \text{if } i = j; \\ 0 & \text{if } i < j; \\ B_Q(\alpha_i^\vee, \alpha_j^\vee) & \text{if } i > j. \end{cases}$$

Recall from Corollary 3.7 that $\mathbf{c} \in \text{Ftn}(i(\bar{\chi}))$ gives rise to a Whittaker functional of $\Theta(\overline{\text{SL}}_n^{(n)}, \bar{\chi})$ if and only if for all y and $\alpha \in \Delta$,

$$\mathbf{c}(\mathbf{s}_{\mathbf{w}_\alpha[y]}) = q^{k_y, \alpha^{-1}} \cdot \mathbf{\Gamma}(y, \alpha^\vee) \cdot \mathbf{g}_{\psi^{-1}}(B(\alpha^\vee, y_\rho))^{-1} \cdot \mathbf{c}(\mathbf{s}_y).$$

For $1 \leq i \leq r$, write $y_{(i)} = \mathbf{w}_{\alpha_i} \mathbf{w}_{\alpha_{i-1}} \dots \mathbf{w}_{\alpha_1}[0]$ and we set $y_{(0)} = 0$. Recall $\mathbf{t}(\mathbf{w}_\alpha, y)$ is the coefficient in the above formula. In this case, it reads $\mathbf{t}(\mathbf{w}_\alpha, y) = q^{k_y, \alpha^{-1}} \cdot \mathbf{\Gamma}(y, \alpha^\vee) \cdot \mathbf{g}_{\psi^{-1}}(\langle y_\rho, \alpha \rangle)^{-1}$ since $Q(\alpha^\vee) = 1$ (and therefore $n_\alpha = n$) for all $\alpha \in \Delta$. Then, in order to have $\dim \text{Wh}_\psi(\Theta(\overline{\text{SL}}_n^{(n)}, \bar{\chi})) = 1$, we must have the equality

$$(14) \quad \bar{\chi}(\mathbf{s}_{y_{Q,n}}) = \mathbf{T}(\mathbf{w}_\natural, 0) \text{ where } \mathbf{T}(\mathbf{w}_\natural, 0) = \prod_{i=1}^r \mathbf{t}(\mathbf{w}_{\alpha_i}, y_{(i-1)}).$$

We would like to show that the equality (14) is also sufficient. Consider any $\mathbf{w}' \in W, y \in \mathcal{O}_0$, one has $\mathbf{c}(\mathbf{s}_{\mathbf{w}'[y]}) = \mathbf{T}(\mathbf{w}', y) \cdot \mathbf{c}(\mathbf{s}_y)$. Now assume $\mathbf{w}'[y] - y \in Y_{Q,n}$, we have

$$\mathbf{c}(\mathbf{s}_{\mathbf{w}'[y]-y+y}) = \bar{\chi}(\mathbf{s}_{\mathbf{w}'[y]-y}) \cdot \mathbf{c}(\mathbf{s}_y) \cdot \varepsilon^{D(\mathbf{w}'[y]-y, y)}.$$

To show $\dim \text{Wh}_\psi(\Theta(\overline{\text{SL}}_n^{(n)}, \bar{\chi})) = 1$, it suffices to show $\mathbf{c}(\mathbf{s}_y)$ to be nonzero for all $y \in \mathcal{O}_0$ such that $\mathbf{w}'[y] - y \in Y_{Q,n}$. That is, it requires

$$(15) \quad \bar{\chi}(\mathbf{s}_{\mathbf{w}'[y]-y}) = \varepsilon^{D(\mathbf{w}'[y]-y, y)} \cdot \mathbf{T}(\mathbf{w}', y).$$

Write $\mathbf{w}'[y] - y = \sum_{i=2}^r k_i \cdot \alpha_{i,Q,n}^\vee + k_1 \cdot y_{Q,n}$. Note \mathcal{O}_0 is $Y_{Q,n}^{\text{sc}}$ -free, thus $k_1 \neq 0$. We may reduce the negative case to the positive case by simple computation, and therefore we can assume $k_1 \geq 1$. Further more, we may apply induction on k_1 , and thus it suffices to: i) prove the inductive step, ii) check the equality (15) when $\mathbf{w}'[y] - y = \sum_{i=2}^r k_i \alpha_{i,Q,n}^\vee + y_{Q,n}$. The assertion i) can be checked easily, and thus we will only outline the proof of ii).

For ii), if $\mathbf{w}'[y] - y = \sum_{i=2}^r k_i \alpha_{i,Q,n}^\vee + y_{Q,n}$, then it is not hard to see $\mathbf{w}'[y] - y = \mathbf{w}(y_{Q,n})$, i.e. $\mathbf{w}^{-1} \mathbf{w}'[y] - \mathbf{w}^{-1} y = y_{Q,n}$ for some $\mathbf{w} \in W$. We may change \mathbf{w} if necessary such that $\mathbf{w}^{-1} y = 0$. With this assumption, $\mathbf{w}^{-1} \mathbf{w}' \mathbf{w} = \mathbf{w}_\natural$, i.e. $\mathbf{w}' = \mathbf{w} \mathbf{w}_\natural \mathbf{w}^{-1}$. Therefore, we are reduced to show that for any $\mathbf{w} \in W$:

$$(16) \quad \bar{\chi}(\mathbf{s}_{\mathbf{w} \mathbf{w}_\natural [0] - \mathbf{w} [0]}) = \varepsilon^{D(\mathbf{w} \mathbf{w}_\natural [0] - \mathbf{w} [0], \mathbf{w} [0])} \cdot \mathbf{T}(\mathbf{w} \mathbf{w}_\natural \mathbf{w}^{-1}, \mathbf{w} [0]).$$

To show (16), we would like to apply induction on the length of \mathbf{w} . When $\mathbf{w} = 1$, it is just the equality (14). For the induction step, assuming the equality (16), we would like to prove that for $\alpha \in \Delta$ the following equality holds:

$$(17) \quad \bar{\chi}(\mathbf{s}_{\mathbf{w}_\alpha \mathbf{w} \mathbf{w}_\natural [0] - \mathbf{w}_\alpha \mathbf{w} [0]}) = \varepsilon^{D(\mathbf{w}_\alpha \mathbf{w} \mathbf{w}_\natural [0] - \mathbf{w}_\alpha \mathbf{w} [0], \mathbf{w}_\alpha \mathbf{w} [0])} \cdot \mathbf{T}(\mathbf{w}_\alpha \mathbf{w} \mathbf{w}_\natural \mathbf{w}^{-1} \mathbf{w}_\alpha^{-1}, \mathbf{w}_\alpha \mathbf{w} [0]).$$

For this purpose, write $x := \mathbb{w}\mathbb{w}_{\mathfrak{h}}[0] - \mathbb{w}[0] \in Y_{Q,n}$. We have $n_{\alpha} | \langle x, \alpha \rangle$. Write $\langle x, \alpha \rangle = k \cdot n_{\alpha}$.

The left hand side of (17) is

$$\begin{aligned} & \overline{\chi}(\mathbf{s}_{x - \langle x, \alpha \rangle \alpha^{\vee}}) \\ &= \overline{\chi}(\mathbf{s}_x) \cdot \overline{\chi}(\mathbf{s}_{-kn_{\alpha}\alpha^{\vee}}) \cdot \epsilon^{D(x, -kn_{\alpha}\alpha^{\vee})} \\ &= \overline{\chi}(\mathbf{s}_x) \cdot \overline{\chi}(\overline{h}_{\alpha}(\varpi^{n_{\alpha}}))^{-k} \\ &= q^k \cdot \overline{\chi}(\mathbf{s}_x). \end{aligned}$$

Meanwhile, the right hand side of (17) is

$$\begin{aligned} & \epsilon^{D(\mathbb{w}_{\alpha}(x), \mathbb{w}_{\alpha}\mathbb{w}[0])} \cdot \mathbf{t}(\mathbb{w}_{\alpha}, \mathbb{w}\mathbb{w}_{\mathfrak{h}}[0]) \cdot \mathbf{T}(\mathbb{w}\mathbb{w}_{\mathfrak{h}}\mathbb{w}^{-1}, \mathbb{w}[0]) \cdot \mathbf{t}(\mathbb{w}_{\alpha}, \mathbb{w}_{\alpha}\mathbb{w}[0]) \\ &= \epsilon^{D(x, \mathbb{w}_{\alpha}\mathbb{w}[0] - \mathbb{w}[0])} \cdot \overline{\chi}(\mathbf{s}_x) \cdot \mathbf{t}(\mathbb{w}_{\alpha}, \mathbb{w}\mathbb{w}_{\mathfrak{h}}[0]) \cdot \mathbf{t}(\mathbb{w}_{\alpha}, \mathbb{w}_{\alpha}\mathbb{w}[0]) \text{ by (16)} \\ &= \epsilon^{D(x, \mathbb{w}_{\alpha}\mathbb{w}[0] - \mathbb{w}[0])} \cdot \overline{\chi}(\mathbf{s}_x) \cdot \mathbf{t}(\mathbb{w}_{\alpha}, \mathbb{w}\mathbb{w}_{\mathfrak{h}}[0]) \cdot \mathbf{t}(\mathbb{w}_{\alpha}, \mathbb{w}[0])^{-1} \\ &= \epsilon^{D(x, \mathbb{w}_{\alpha}\mathbb{w}[0] - \mathbb{w}[0])} \cdot q^{\left\lceil \frac{\langle \mathbb{w}[0], \alpha \rangle}{n_{\alpha}} \right\rceil - 1 + k} \cdot \mathbf{g}_{\psi^{-1}}(\langle \mathbb{w}(0_{\rho}), \alpha \rangle Q(\alpha^{\vee}))^{-1} \cdot \epsilon^{\langle \mathbb{w}(0_{\rho}), \alpha \rangle \cdot D(\mathbb{w}\mathbb{w}_{\mathfrak{h}}[0], \alpha^{\vee})} \\ & \quad \cdot \overline{\chi}(\mathbf{s}_x) \cdot q^{-\left\lceil \frac{\langle \mathbb{w}[0], \alpha \rangle}{n_{\alpha}} \right\rceil + 1} \cdot \mathbf{g}_{\psi^{-1}}(\langle \mathbb{w}(0_{\rho}), \alpha \rangle Q(\alpha^{\vee})) \cdot \epsilon^{\langle \mathbb{w}(0_{\rho}), \alpha \rangle \cdot D(\mathbb{w}[0], \alpha^{\vee})} \\ &= \overline{\chi}(\mathbf{s}_x) \cdot q^k \cdot \epsilon^{\langle \mathbb{w}(0_{\rho}), \alpha \rangle D(x, \alpha^{\vee})} \cdot \epsilon^{\langle \mathbb{w}(0_{\rho}), \alpha \rangle D(x, \alpha^{\vee})} \\ &= \overline{\chi}(\mathbf{s}_x) \cdot q^k, \end{aligned}$$

which is clearly equal to the left hand side. To summarize, we have

Proposition 4.1. *Let $\overline{\chi} \in \text{Hom}_k(Z(\overline{T}), \mathbf{C}^{\times})$ be an exceptional character of $\overline{SL}_n^{(n)}$. Then*

$$\dim \text{Wh}_{\psi}(\Theta(\overline{SL}_n^{(n)}, \overline{\chi})) = 1$$

if and only if $\overline{\chi}$ is the unique exceptional character satisfying (14).

We would like to explicate the condition given by (14). Note $\mathbf{T}(\mathbb{w}_{\mathfrak{h}}, 0) = \prod_{i=1}^r \mathbf{t}(\mathbb{w}_{\alpha_i}, y_{\langle i-1 \rangle})$.

Lemma 4.2. *One has*

$$\mathbf{T}(\mathbb{w}_{\mathfrak{h}}, 0) = \begin{cases} q^{-r/2} & \text{if } n \text{ is odd;} \\ \epsilon^{n(n-2)/8} \cdot q^{-n/2} \cdot \mathbf{g}_{\psi^{-1}}(-n/2)^{-1} & \text{if } n \text{ is even.} \end{cases}$$

Proof. We compute each $\mathbf{t}(\mathbb{w}_{\alpha_i}, y_{\langle i-1 \rangle})$ for $1 \leq i \leq r$. First, one can check easily $y_{\langle i \rangle} = \sum_{j=1}^i i \cdot \alpha_j^{\vee} = \alpha_1^{\vee} + 2\alpha_2^{\vee} + \dots + i \cdot \alpha_i^{\vee}$. Thus, $\langle y_{\langle i-1 \rangle}, \alpha_i \rangle = -(i-1)$ and therefore

$$k_{y_{\langle i-1 \rangle}, \alpha_i} = 0 \text{ for all } 1 \leq i \leq r.$$

Second, $\mathbf{\Gamma}(y_{\langle i-1 \rangle}, \alpha_i^{\vee}) = \epsilon^{-i \cdot D(y_{\langle i-1 \rangle}, \alpha_i^{\vee})}$. Since $D(\alpha_j^{\vee}, \alpha_i^{\vee}) = 0$ for all $j < i$, we see $\mathbf{\Gamma}(y_{\langle i-1 \rangle}, \alpha_i^{\vee}) = 1$. Therefore, $\mathbf{t}(\mathbb{w}_{\alpha_i}, y_{\langle i-1 \rangle}) = q^{-1} \cdot \mathbf{g}_{\psi^{-1}}(-i)^{-1}$. Now, if $1 \leq i, j \leq n$ and $i + j = n$, one has

$$\begin{aligned} & \mathbf{g}_{\psi^{-1}}(-i)^{-1} \cdot \mathbf{g}_{\psi^{-1}}(-j)^{-1} \\ &= \mathbf{g}_{\psi^{-1}}(-i)^{-1} \cdot \overline{(\mathbf{g}_{\psi^{-1}}(j) \cdot \epsilon^j)}^{-1} \\ &= |\mathbf{g}_{\psi^{-1}}(j)|^{-2} \cdot \epsilon^i \\ &= q \cdot \epsilon^i. \end{aligned}$$

The result then follows from simply multiplying together each term. \square

4.2.2. Interlude: Weil-index. Let ψ be a nontrivial additive character of F . Let γ_{ψ} be the Weil-index satisfying

$$\gamma_{\psi}(b^2) = 1, \gamma_{\psi}(b)^2 = (b, b)_2, \gamma_{\psi}(bc) = \gamma_{\psi}(b)\gamma_{\psi}(c) \cdot (b, c)_2.$$

For any $a \in F^{\times}$, let $\psi_a : x \mapsto \psi(ax)$ be the twisted additive character. Then

$$\gamma_{\psi_a}(b) = \gamma_{\psi}(b) \cdot (a, b)_2.$$

Lemma 4.3. *Suppose $n = 2m$ is an even number. Then the following equality holds:*

$$\mathbf{g}_{\psi^{-1}}(m) = \frac{q^{-1/2}}{\gamma_{\psi}(\varpi)}.$$

Proof. By definition, $\mathbf{g}_{\psi^{-1}}(m)$ is equal to

$$\begin{aligned} & \int_{O_F^\times} (u, \varpi)_2 \cdot \psi^{-1}(\varpi^{-1}u) \mu(u) \\ &= \int_{O_F^\times} \gamma_{\psi}(\varpi u) \gamma_{\psi}(\varpi)^{-1} \gamma_{\psi}(u)^{-1} \cdot \psi^{-1}(\varpi^{-1}u) \mu(u) \\ &= \gamma_{\psi}(\varpi)^{-1} \cdot \int_{O_F^\times} \gamma_{\psi}(\varpi u) \cdot \psi^{-1}(\varpi^{-1}u) \mu(u). \end{aligned}$$

However, by Equality (3.7) of [Szp1, Lemma 3.2],

$$\gamma_{\psi}(\varpi u) = q^{-1/2} \left(1 + q \int_{O_F^\times} \psi(\varpi^{-1}v^2u) \mu(v) \right).$$

Thus,

$$\begin{aligned} & \mathbf{g}_{\psi^{-1}}(m) \\ &= q^{-1/2} \cdot \gamma_{\psi}(\varpi)^{-1} \cdot \int_{O_F^\times} \left(1 + q \int_{O_F^\times} \psi(\varpi^{-1}v^2u) \mu(v) \right) \psi^{-1}(\varpi^{-1}u) \mu(u) \\ &= q^{-1/2} \cdot \gamma_{\psi}(\varpi)^{-1} \cdot \left(-1/q + q \cdot \int_{O_F^\times} \int_{O_F^\times} \psi(\varpi^{-1}u(v^2 - 1)) \mu(u) \mu(v) \right) \end{aligned}$$

Let $D = \{v \in O_F^\times : |1 - v^2| = 1\}$ and $H = \{v \in O_F^\times : |1 - v^2| \leq q^{-1}\}$. We get

$$\begin{aligned} & \int_{O_F^\times} \left(\int_{O_F^\times} \psi(\varpi^{-1}u(v^2 - 1)) \mu(u) \right) \mu(v) \\ &= \int_{v \in H} \int_{O_F^\times} \psi(\varpi^{-1}u(v^2 - 1)) \mu(u) \mu(v) + \int_{v \in D} \int_{O_F^\times} \psi(\varpi^{-1}u(v^2 - 1)) \mu(u) \mu(v) \\ &= \mu(H) \cdot (1 - q^{-1}) + \mu(D) \cdot (-q^{-1}) \text{ by (8.19) of [Szp1, Lemma 8.6]} \\ &= 2q^{-1} \cdot (1 - q^{-1}) + (1 - 3q^{-1}) \cdot (-q^{-1}) \text{ by [Szp1, Lemma 8.9]} \\ &= q^{-1} + q^{-2}. \end{aligned}$$

The result follows easily by simplification. \square

4.2.3. An explicit criterion. Consider the *unitary* distinguished character $\overline{\chi}_{\psi'}^0$ constructed in [GG], which we recalled and gave in formula (5). Then the character $\overline{\chi}_{\psi'} = \overline{\chi}_{\psi'}^0 \cdot \delta_B(\cdot)^{\frac{1}{2n}}$ is an exceptional character. Note in the simply-connected case, $J = Y_{Q,n}^{\text{sc}}$. For the definition of $\overline{\chi}_{\psi'}^0$, we pick a basis $\{y_i\}$ for $Y_{Q,n}$ such that $\{k_i y_i\}$ is a basis for $J = Y_{Q,n}^{\text{sc}}$. Then by definition,

$$\overline{\chi}_{\psi'}^0(\mathbf{s}_{y_i}) = \gamma_{\psi'}(\varpi)^{2(k_i-1)Q(y_i)/n}$$

and, for $y = \sum_i n_i y_i \in Y_{Q,n}$, one has

$$\overline{\chi}_{\psi'}^0(\mathbf{s}_y) = \prod_i \overline{\chi}_{\psi'}^0(\varpi^{n_i})^{2(k_i-1)Q(y_i)/n} \cdot \epsilon^{\sum_{i < j} n_i n_j D(y_i, y_j)}.$$

For the covering group $\overline{\text{SL}}_n^{(n)}$, we take $y_i = n\alpha_i^\vee$, $2 \leq i \leq r$ and $y_1 = y_{Q,n}$, with $k_i = 1$ for $2 \leq i \leq r$ and $k_1 = n$.

Easy computation shows $Q(y_{Q,n}) = r(r+1)/2$, and thus

$$(18) \quad \begin{aligned} & \overline{\chi}_{\psi'}(\mathbf{s}_{y_{Q,n}}) \\ &= \overline{\chi}_{\psi'}^0(\mathbf{s}_{y_{Q,n}}) \cdot \delta_B(\mathbf{s}_{y_{Q,n}})^{\frac{1}{2n}} \\ &= \gamma_{\psi'}(\varpi)^{(n-1)^2} \cdot q^{-(n-1)/2}. \end{aligned}$$

Proposition 4.4. *For the exceptional character $\overline{\chi}_{\psi'} = \overline{\chi}_{\psi'}^0 \cdot \delta_B(\cdot)^{\frac{1}{2n}}$ given above, one has $\dim \text{Wh}_{\psi}(\Theta(\overline{SL}_n^{(n)}, \overline{\chi}_{\psi'})) = 1$ if and only if the following holds:*

$$\begin{cases} \text{any } \psi', & \text{if } n \text{ is odd;} \\ \gamma_{\psi'}(\varpi) = \gamma_{\psi}(\varpi), & \text{if } n \equiv 0, 2 \pmod{8}; \\ \gamma_{\psi'}(\varpi) = (-1, \varpi)_4 \cdot \gamma_{\psi}(\varpi), & \text{if } n \equiv 4 \pmod{8}; \\ \gamma_{\psi'}(\varpi) = \gamma_{\psi}(\varpi)^{-1}, & \text{if } n \equiv 6 \pmod{8}. \end{cases}$$

Proof. By the value of $\overline{\chi}_{\psi'}(\mathbf{s}_{y_{Q,n}})$ in (18), it follows from Lemma 4.2 that the equality (14) is equivalent to

$$(19) \quad \gamma_{\psi'}(\varpi)^{(n-1)^2} \cdot q^{-(n-1)/2} = \begin{cases} q^{-r/2} & \text{if } n \text{ is odd;} \\ (-1, \varpi)_n^{n(n-2)/8} \cdot q^{-n/2} \cdot \mathbf{g}_{\psi^{-1}}(-n/2)^{-1} & \text{if } n \text{ is even.} \end{cases}$$

For n odd, the equality holds for any ψ' . Now we assume n even.

For $n = 4k + 2$, by Lemma 4.3, the required equality in (19) becomes

$$\gamma_{\psi'}(\varpi) = \gamma_{\psi}(\varpi)^{2k+1}.$$

In particular, if k is even, it is equivalent to $\gamma_{\psi'}(\varpi) = \gamma_{\psi}(\varpi)$. If k is odd, it is equivalent to $\gamma_{\psi'}(\varpi) = \gamma_{\psi}(\varpi)^{-1}$.

For $n = 4k$, applying again Lemma 4.3, the equality in (19) reads

$$\gamma_{\psi'}(\varpi) = (-1, \varpi)_n^k \cdot \gamma_{\psi}(\varpi) = (-1, \varpi)_4 \cdot \gamma_{\psi}(\varpi).$$

A special case is when k is even. In this case $(-1, \varpi)_4 = 1$ and therefore it is equivalent to $\gamma_{\psi'}(\varpi) = \gamma_{\psi}(\varpi)$. \square

Corollary 4.5. *Consider $\psi' = \psi_a$ for some $a \in F^\times$. Assume that ψ_a has conductor O_F , i.e. $a \in O_F^\times$. Then $\dim \text{Wh}_{\psi}(\Theta(\overline{SL}_n^{(n)}, \overline{\chi}_{\psi_a})) = 1$ if and only if the following holds:*

$$\begin{cases} a \in O_F^\times, & \text{if } n \text{ is odd;} \\ a \in (O_F^\times)^2, & \text{if } n \equiv 0, 2 \pmod{8}; \\ a^2 \in -(O_F^\times)^4, & \text{if } n \equiv 4 \pmod{8}; \\ a \in -(O_F^\times)^2, & \text{if } n \equiv 6 \pmod{8}. \end{cases}$$

Example 4.6. The first nontrivial example is the metaplectic covering $\overline{SL}_2^{(2)}$. For the character ψ_a , the representation $\Theta(\overline{SL}_2^{(2)}, \overline{\chi}_{\psi_a})$ is the even Weil representation in the following exact sequence:

$$\text{St}(\overline{\chi}_{\psi_a}) \hookrightarrow I(\overline{\chi}_{\psi_a}) \twoheadrightarrow \Theta(\overline{SL}_2^{(2)}, \overline{\chi}_{\psi_a}),$$

where $\text{St}(\overline{\chi}_{\psi_a})$ is the metaplectic analogue of the Steinberg representation. Corollary 4.5 above recovers the well-known fact that for $\overline{SL}_2^{(2)}$ the even Weil representation $\Theta(\overline{SL}_2^{(2)}, \overline{\chi}_{\psi_a})$ (for unramified data) is ψ -generic if and only if $a \in (O_F^\times)^2$.

Example 4.7. We also make explicit the example of $\overline{SL}_3^{(3)}$. Consider $\overline{SL}_3^{(3)}$ with cocharacter lattice $Y = \langle \alpha_1^\vee, \alpha_2^\vee \rangle$. Consider Q such that $Q(\alpha_i^\vee) = 1$. Then

$$Y_{Q,n} = \langle 2\alpha_1^\vee + \alpha_2^\vee, 3\alpha_1^\vee \rangle = \langle 2\alpha_2^\vee + \alpha_1^\vee, 3\alpha_2^\vee \rangle.$$

Note $Y = \langle 2\alpha_1^\vee + \alpha_2^\vee, \alpha_1^\vee \rangle = \langle 2\alpha_2^\vee + \alpha_1^\vee, \alpha_2^\vee \rangle$. We know $\rho = \alpha_1^\vee + \alpha_2^\vee$. For $y = 0$ one has

$$y_\rho = 0_\rho = -(\alpha_1^\vee + \alpha_2^\vee).$$

Consider $w_{\mathfrak{h}} = w_{\alpha_1} w_{\alpha_2}$, then $w_{\alpha_2}[y] = \alpha_2^\vee$ and moreover $w_{\alpha_1} w_{\alpha_2}[y] = 2\alpha_1^\vee + \alpha_2^\vee$. One has

$$\begin{aligned} & \mathbf{c}(s_{w_1 w_2}[y]) \\ &= q^{k_{w_2[y], \alpha_1} - 1} \cdot \Gamma(w_2[y], \alpha_1^\vee) \cdot \mathbf{g}_{\psi^{-1}}(Q(\alpha_1^\vee)(\langle w_2[y], \alpha_1 \rangle - 1))^{-1} \\ & \quad \cdot q^{k_{y, \alpha_2} - 1} \cdot \Gamma(y, \alpha_2^\vee) \cdot \mathbf{g}_{\psi^{-1}}(Q(\alpha_2^\vee)(\langle y, \alpha_2 \rangle - 1))^{-1} \cdot \mathbf{c}(s_y) \\ &= q^{\left\lceil \frac{\langle \alpha_2^\vee, \alpha_1 \rangle}{3} \right\rceil + \left\lceil \frac{\langle y, \alpha_2 \rangle}{3} \right\rceil - 2} \cdot \Gamma(\alpha_2^\vee, \alpha_1^\vee) \cdot \Gamma(0, \alpha_2^\vee) \cdot \mathbf{g}_{\psi^{-1}}(-2)^{-1} \mathbf{g}_{\psi^{-1}}(-1)^{-1} \cdot \mathbf{c}(1_{\overline{\text{SL}}_3^{(3)}}) \\ &= q^{-2} \cdot q \cdot \mathbf{c}(1_{\overline{\text{SL}}_3^{(3)}}) = q^{-1}, \end{aligned}$$

where \mathbf{c} is normalized to take value 1 at the $1 \in \overline{\text{SL}}_3^{(3)}$. This implies that necessarily $\mathbf{c}(s_{w_1 w_2}[y]) = q^{-1}$, and thus

$$\overline{\chi}(s_{w_1 w_2}[y]) = q^{-1}.$$

Note, this is not a consequence of $\overline{\chi}$ being exceptional, although it is compatible. Clearly, an exceptional character $\overline{\chi}$ is such that

$$\begin{cases} \overline{\chi}(s_{w_1 w_2}[y])^3 = q^{-3}, \\ \overline{\chi}(s_{3\alpha_1^\vee}) = q^{-1}. \end{cases}$$

In particular, if $\overline{\chi}(s_{w_1 w_2}[y]) = \zeta \cdot q^{-1}$ for some third root of unity $\zeta \neq 1$, then $\dim \text{Wh}_\psi(\Theta(\overline{\text{SL}}_3^{(3)}, \overline{\chi})) = 0$ for such $\overline{\chi}$.

4.3. Case III: $\overline{\text{SL}}_{r+1}^{(n)}$, $n = r + 2$. For $n = r + 2$, we show $Y_{Q,n} = Y_{Q,n}^{\text{sc}}$ and therefore Corollary 3.15 applies. Pick any $(y_1, y_2, \dots, y_{r+1}) \in Y_{Q,n}$, we have

$$a \equiv y_1 \equiv y_2 \equiv \dots \equiv y_{r+1} \pmod{n},$$

where $a \in \{0, 1, 2, \dots, r + 1\}$. Write $y_i = k_i n + a$. Since $\sum_{i=1}^{r+1} y_i = 0$, one has

$$n \cdot \left(\sum_{i=1}^{r+1} k_i \right) + (r + 1) \cdot a = 0.$$

In particular, $n \mid (r + 1)a$. However, $\gcd(n, r + 1) = 1$, therefore $n \mid a$ and $a = 0$. That is, $Y_{Q,n} = Y_{Q,n}^{\text{sc}}$ and therefore $\dim \text{Wh}_\psi(\Theta(\overline{\text{SL}}_{r+1}^{(r+2)}, \overline{\chi})) = |\wp_{Q,n}(\mathcal{O}_{Q,n}^F)|$. Note, the equality $Y_{Q,n} = Y_{Q,n}^{\text{sc}}$ reflects the fact that the dual group for $\overline{\text{SL}}_n^{(n+1)}$ is PGL_n (cf. [We2, §2.7.2]).

We claim that the dimension is equal to 1 in this case. Let $\mathcal{O}_y \in \mathcal{O}_{Q,n}^F$ be a $Y_{Q,n}^{\text{sc}}$ -free orbit with $\mathbf{i}_A^*(y_\rho) = (0, 1, \dots, r - 1, r) \in \bigoplus_{i=1}^{r+1} \mathbf{Z}e_i$. We know \mathcal{O}_y is $Y_{Q,n}$ -free (or equally, $Y_{Q,n}^{\text{sc}}$ -free). Moreover, one can check easily $\wp_{Q,n}(\mathcal{O}_{Q,n}^F) = \{\wp_{Q,n}(\mathcal{O}_y)\}$. Therefore $\dim \text{Wh}_\psi(\Theta(\overline{\text{SL}}_{r+1}^{(r+2)}, \overline{\chi})) = 1$ for the unique exceptional character $\overline{\chi}$ in this case.

4.4. Case IV: $\overline{\text{SL}}_{r+1}^{(n)}$, $n \geq r + 3$.

Lemma 4.8. *Consider $y \in Y$ such that $\mathbf{i}_A^*(y_\rho) = (x_1^*, x_2^*, \dots, x_r^*, x_{r+1}^*)$ with $x_i^* = i - 1$. If $n \geq r + 3$, the orbit \mathcal{O}_y is $Y_{Q,n}$ -free.*

Proof. Suppose not, there exists $w \neq 1$ such that $w[y] - y \in Y_{Q,n}$. Identify w with a permutation, then we have

$$(x_1^*, x_2^*, \dots, x_{r+1}^*) - (x_{w(1)}^*, x_{w(2)}^*, \dots, x_{w(r+1)}^*) \in Y_{Q,n}.$$

More precisely, $i - w(i) \equiv j - w(j) \pmod{n}$ for all i, j . Clearly, $n \nmid (i - w(i))$ for all i , otherwise one can deduce $w(i) = i$ for all i and therefore $w = 1$. That is, $(i - w(i))$ is either negative or positive. We reorder the terms $(i - w(i))$'s as

$$-r \leq (i_1 - w(i_1)) \leq (i_2 - w(i_2)) \leq \dots < 0 < \dots \leq (i_r - w(i_r)) \leq (i_{r+1} - w(i_{r+1})) \leq r.$$

Write $(i_1 - \mathbf{w}(i_1)) = -s, s \in \mathbf{N}$ and $(i_{r+1} - \mathbf{w}(i_{r+1})) = t, t \in \mathbf{N}$. It is easy to see that any negative $i - \mathbf{w}(i)$ must be equal to $-s$, and any positive $i - \mathbf{w}(i)$ must be equal to t .

We claim that $2 < t + s \leq r + 1$ and therefore $n \nmid (t + s)$, i.e. $\mathbf{w}[y] - y \notin Y_{Q,n}$ for all $\mathbf{w} \neq 1$. Note $0 - \mathbf{w}(0) = -s$ and $r - \mathbf{w}(r) = t$. Suppose $t + s > r + 1$, then there exists i_0 such that $r + 1 - t < i_0 < 1 + s$. However, there exists no i' such that $\mathbf{w}(i') = i_0$. This is a contradiction, and the claim follows.

Therefore \mathcal{O}_y is $Y_{Q,n}$ -free for the given y . The proof is completed. \square

It follows that $\dim \text{Wh}_\psi(\Theta(\overline{\text{SL}}_{r+1}^{(n)}, \bar{\chi})) \geq 1$ for $n \geq r + 3$. In principle, one could proceed as in §4.2 to analyze every element in $\wp_{Q,n}(\mathcal{O}_{Q,n,\text{sc}}^F)$ and determine completely $\dim \text{Wh}_\psi(\Theta(\overline{\text{SL}}_{r+1}^{(n)}, \bar{\chi}))$ in this case. However, the level of complexity of the computation depend inevitably on (the center of) the dual group of $\overline{\text{SL}}_r^{(n)}$ and could be quite involved for general $n \geq r + 3$.

We summarize for the $n \leq r + 2$ cases below.

Theorem 4.9. *Consider the Brylinski-Deligne covering $\overline{\text{SL}}_{r+1}^{(n)}, n \leq r + 2$ with $Q(\alpha^\vee) = 1$ for all coroots α^\vee . Let $\bar{\chi}$ be an exceptional character of $\overline{\text{SL}}_{r+1}^{(n)}$. Then $\dim \text{Wh}_\psi(\Theta(\overline{\text{SL}}_{r+1}^{(n)}, \bar{\chi})) = 1$ if and only if*

- $n = r + 2$ and $\bar{\chi}$ is the only exceptional character, or
- $n = r + 1$ and $\bar{\chi}$ is the unique exceptional character satisfying (14).

5. THE $C_r, r \geq 2$ CASE

Consider the Dynkin diagram for the simple coroots for C_r :

$$\alpha_1^\vee \text{---} \alpha_2^\vee \text{---} \cdots \text{---} \alpha_{r-2}^\vee \text{---} \alpha_{r-1}^\vee \text{---} \alpha_r^\vee$$

Let $Y = Y^{\text{sc}} = \langle \alpha_1^\vee, \alpha_2^\vee, \dots, \alpha_{r-1}^\vee, \alpha_r^\vee \rangle$ be the cocharacter lattice of Sp_{2r} , where α_r^\vee is the short coroot. Let Q be the Weyl-invariant quadratic on Y such such $Q(\alpha_r^\vee) = 1$. Then the bilinear form B_Q is given by

$$B_Q(\alpha_i^\vee, \alpha_j^\vee) = \begin{cases} 2 & \text{if } i = j = r; \\ 4 & \text{if } 1 \leq i = j \leq r - 1; \\ -2 & \text{if } j = i + 1; \\ 0 & \text{if } \alpha_i^\vee, \alpha_j^\vee \text{ are not adjacent.} \end{cases}$$

Simple computation gives:

$$Y_{Q,n} = \left\{ \sum_{i=1}^n x_i \alpha_i^\vee : n \mid (2x_i) \right\}.$$

We write $n_2 := n/\gcd(2, n)$. Then

$$Y_{Q,n} = \langle n_2 \alpha_1^\vee, n_2 \alpha_2^\vee, \dots, n_2 \alpha_{r-1}^\vee, n_2 \alpha_r^\vee \rangle$$

and

$$Y_{Q,n}^{\text{sc}} = \langle n_2 \alpha_1^\vee, n_2 \alpha_2^\vee, \dots, n_2 \alpha_{r-1}^\vee, n \alpha_r^\vee \rangle.$$

The map $\mathbf{i}_C : \bigoplus_{i=1}^r \mathbf{Z} \alpha_i^\vee \rightarrow \bigoplus_{i=1}^r \mathbf{Z} e_i$ is given by

$$\mathbf{i}_C : (x_1, x_2, x_3, \dots, x_r) \mapsto (x_1, x_2 - x_1, x_3 - x_2, \dots, x_{r-1} - x_{r-2}, x_r - x_{r-1}).$$

Note \mathbf{i}_C is an isomorphism. The Weyl group is $W = S_r \rtimes (\mathbf{Z}/2\mathbf{Z})^r$, where S_r is the permutation group on $\bigoplus_i \mathbf{Z} e_i$ and each $(\mathbf{Z}/2\mathbf{Z})_i$ acts by $e_i \mapsto \pm e_i$. In particular, $\mathbf{w}_{\alpha_i}, 1 \leq i \leq r - 1$, acts on $(y_1, y_2, \dots, y_r) \in \bigoplus_i \mathbf{Z} e_i$ by exchanging y_i and y_{i+1} , while \mathbf{w}_{α_r} acts by (-1) on $\mathbf{Z} e_r$.

Moreover, $y \in Y$ lies in $Y_{Q,n}$ if and only all entries of $\mathbf{i}_C(y)$ are divisible by n_2 . It is easy to obtain

$$Y_{Q,n} = \left\{ \begin{array}{l} (y_1, y_2, \dots, y_r) \in \bigoplus_{i=1}^r \mathbf{Z}e_i : \\ \bullet \ n_2 | y_i \text{ for all } i. \end{array} \right\} \text{ and } Y_{Q,n}^{\text{sc}} = \left\{ \begin{array}{l} (y_1, y_2, \dots, y_r) \in \bigoplus_{i=1}^r \mathbf{Z}e_i : \\ \bullet \ n_2 | y_i \text{ for all } i, \\ \bullet \ n | \sum_i y_i. \end{array} \right\}$$

We further note

$$2\rho = \sum_{i=1}^r (2r - 2i + 1)e_i = \sum_{i=1}^r i(2r - i)\alpha_i^\vee.$$

Assume $x_0 = 0$, then

$$\mathbf{i}_C(y_\rho) = (x_i - x_{i-1} - (r - i + 1/2))_{1 \leq i \leq r}.$$

Write $x_i^* := x_i - x_{i-1} - (r - i)$, and also $\mathbf{i}_C^*(y_\rho) := (x_1^*, x_2^*, \dots, x_{r-1}^*, x_r^*)$. Then

$$\mathbf{i}_C(y_\rho) = \mathbf{i}_C^*(y_\rho) - (1/2, 1/2, \dots, 1/2, 1/2).$$

We will discuss the two cases depending on the parity of n separately.

5.1. For n odd. In this case, $n_2 = n$ and

$$nY = Y_{Q,n}^{\text{sc}} = Y_{Q,n} = \left\{ (y_1, \dots, y_r) \in \bigoplus_{i=1}^r \mathbf{Z}e_i : n | y_i \text{ for all } i \right\}.$$

The complex dual group for $\overline{\text{Sp}}_{2r}^{(n)}$ for n odd is SO_{2r+1} .

Proposition 5.1. *Let n be an odd number, one has*

$$\left\{ \begin{array}{ll} |\wp_{Q,n}(\mathcal{O}_{Q,n,\text{sc}}^F)| \geq 2 & \text{if } n \geq 2r + 3; \\ |\wp_{Q,n}(\mathcal{O}_{Q,n,\text{sc}}^F)| = 1 & \text{if } n = 2r + 1; \\ |\wp_{Q,n}(\mathcal{O}_{Q,n,\text{sc}}^F)| = 0 & \text{if } n \leq 2r - 1. \end{array} \right.$$

Therefore, when n is odd, we have $\dim \text{Wh}_\psi(\Theta(\overline{\text{Sp}}_{2r}^{(n)}, \overline{\chi})) = 1$ if and only if $n = 2r + 1$ for the only exceptional character of $\overline{\text{Sp}}_{2r}^{(2r+1)}$.

Proof. We have written

$$\mathbf{i}_C(y_\rho) = \mathbf{i}_C^*(y_\rho) - (1/2, 1/2, \dots, 1/2, 1/2).$$

Since x_1, \dots, x_r are arbitrary, the associated x_i^* 's are also arbitrary.

First, when $n \geq 2r + 3$, consider the orbits \mathcal{O}_y and $\mathcal{O}_{y'}$ where

$$\mathbf{i}_C^*(y_\rho) = (1, 2, \dots, r - 1, r) \text{ and } \mathbf{i}_C^*(y'_\rho) = (1, 2, \dots, r - 1, r + 1).$$

If $r = 2$, consider \mathcal{O}_y and $\mathcal{O}_{y'}$ with $\mathbf{i}_C^*(y_\rho) = (1, 2)$ and $\mathbf{i}_C^*(y'_\rho) = (1, 3)$. Both $\mathcal{O}_y, \mathcal{O}_{y'}$ are $Y_{Q,n}$ -free orbits. For example, for \mathcal{O}_y , this follows from the fact that the entries of $\mathbf{i}_C(\mathbf{w}(y_\rho)) - \mathbf{i}_C(y_\rho)$ are either $j - i$ or $j + i - 1$, for $0 \leq i, j \leq r - 1$. One can check also $\wp_{Q,n}(\mathcal{O}_y) \neq \wp_{Q,n}(\mathcal{O}_{y'})$, and therefore $|\wp_{Q,n}(\mathcal{O}_{Q,n}^F)| \geq 2$.

Second, assume $n = 2r + 1$. Consider \mathcal{O}_y such that $\mathbf{i}_C^*(y_\rho) = (1, 2, \dots, r - 1, r)$. For $r = 2$, consider $\mathbf{i}_C^*(y_\rho) = (1, 2)$. It can be checked easily that $\wp_{Q,n}(\mathcal{O}_{Q,n}^F) = \{\wp_{Q,n}(\mathcal{O}_y)\}$. Thus, $\dim \text{Wh}_\psi(\Theta(\overline{\text{Sp}}_{2r}^{(2r+1)}, \overline{\chi})) = 1$.

Third, assume $n \leq 2r - 1$, we want to show $\mathcal{O}_{Q,n,\text{sc}}^F = \emptyset$. If $\mathbf{i}_C^*(y_\rho) = (x_1^*, x_2^*, \dots, x_i^*, \dots, x_r^*)$ is such that $x_i^* \equiv x_j^* \pmod{n}$ for some $i \neq j$, then clearly $\mathcal{O}_y \notin \mathcal{O}_{Q,n,\text{sc}}^F$. Now if $n \nmid (x_i^* - x_j^*)$ for all $i \neq j$; since $n \leq 2r - 1$, it is not hard to see that there always exist i, j such that $n | (x_j^* - 1/2) + (x_i^* - 1/2)$, i.e. $n | (x_j^* + x_i^* - 1)$. In this case, one also has $\mathcal{O}_y \notin \mathcal{O}_{Q,n,\text{sc}}^F$. In any case, $\mathcal{O}_{Q,n,\text{sc}}^F = \emptyset$ for $n \leq 2r - 1$.

The proof is completed. \square

5.2. **For n even.** Write $n = 2m$, in this case,

$$Y_{Q,n} = \langle m\alpha_1^\vee, m\alpha_2^\vee, \dots, m\alpha_{r-1}^\vee, m\alpha_r^\vee \rangle, \quad Y_{Q,n}^{\text{sc}} = \langle m\alpha_1^\vee, m\alpha_2^\vee, \dots, m\alpha_{r-1}^\vee, n\alpha_r^\vee \rangle.$$

Equivalently, one has:

$$Y_{Q,n} = \left\{ \begin{array}{l} (y_1, y_2, \dots, y_r) \in \bigoplus_{i=1}^r \mathbf{Z}e_i : \\ \bullet \ m|y_i \text{ for all } i. \end{array} \right\} \text{ and } Y_{Q,n}^{\text{sc}} = \left\{ \begin{array}{l} (y_1, y_2, \dots, y_r) \in \bigoplus_{i=1}^r \mathbf{Z}e_i : \\ \bullet \ m|y_i \text{ for all } i, \\ \bullet \ n|\sum_i y_i. \end{array} \right\}$$

The dual group for $\overline{\text{Sp}}_{2r}^{(n)}$ with n even is Sp_{2r} .

5.2.1. *For $m \geq 2r + 2$.* In this case, consider the orbits $\mathcal{O}_y, \mathcal{O}_{y'}$ given in the proof of Proposition 5.1. They are $Y_{Q,n}$ -free; moreover, \mathcal{O}_y and $\mathcal{O}_{y'}$ are distinct in the image of $\wp_{Q,n}$. Thus, we have $|\wp_{Q,n}(\mathcal{O}_{Q,n}^F)| \geq 2$.

5.2.2. *For $m \leq 2r - 2$.* We can check easily in this case $\mathcal{O}_{Q,n,\text{sc}}^F = \emptyset$.

5.2.3. *For $m = 2r - 1$.* In this case, consider y with $\mathbf{i}_C^*(y_\rho) = (1, 2, \dots, r-1, r)$, i.e.

$$\mathbf{i}_C(y_\rho) = (1 - 1/2, 2 - 1/2, \dots, (r-1) - 1/2, r - 1/2).$$

Consider $\mathbf{w}_{\alpha_r} \in W$, then $\mathbf{i}_C(\mathbf{w}_{\alpha_r}(y_\rho)) = (1 - 1/2, 2 - 1/2, \dots, (r-1) - 1/2, -(r-1/2))$.

Note, \mathcal{O}_y is $Y_{Q,n}^{\text{sc}}$ -free, and $\wp_{Q,n}^{\text{sc}}(\mathcal{O}_{Q,n,\text{sc}}^F) = \{\wp_{Q,n}^{\text{sc}}(\mathcal{O}_y)\} = \{\wp_{Q,n}^{\text{sc}}(\mathcal{O}_0)\}$. However, it is not $Y_{Q,n}$ -free, since $\mathbf{i}_C(y_\rho - \mathbf{w}_{\alpha_r}(y_\rho)) = (0, 0, \dots, m) \in Y_{Q,n}$. Recall that any $\mathbf{c} \in \text{Ftn}(i(\overline{\chi}))$ which gives rise to $\lambda_{\overline{\mathbf{c}}} \in \text{Wh}_\psi(\Theta(\overline{G}, \overline{\chi}))$ satisfies $\mathbf{c}(\mathbf{s}_{\mathbf{w}_{\alpha_r}[y]}) = \mathbf{t}(\mathbf{w}_{\alpha_r}, y) \cdot \mathbf{c}(\mathbf{s}_y)$ where

$$\mathbf{t}(\mathbf{w}_{\alpha_r}, y) = q^{k_{y, \alpha_r} - 1} \cdot \mathbf{\Gamma}(y, \alpha_r^\vee) \cdot \mathbf{g}_{\psi^{-1}}(Q(\alpha_r^\vee) \cdot \langle y_\rho, \alpha_r \rangle)^{-1}.$$

Meanwhile, in our case $\mathbf{w}_{\alpha_r}[y] - y = (-m)\alpha_r^\vee \in Y_{Q,n}$. It follows

$$\mathbf{c}(\mathbf{s}_{\mathbf{w}_{\alpha_r}[y]}) = \varepsilon^{D(\mathbf{w}_{\alpha_r}(y_\rho) - y_\rho, y)} \cdot \overline{\chi}(\mathbf{s}_{\mathbf{w}_{\alpha_r}(y_\rho) - y_\rho}) \cdot \mathbf{c}(\mathbf{s}_y).$$

For \mathbf{c} to be nonzero on \mathcal{O}_y , i.e., $\wp_{Q,n}(\mathcal{O}_y)$ contributes to the right hand side of (10), one has

$$\overline{\chi}(\mathbf{s}_{-m\alpha_r^\vee}) = q^{k_{y, \alpha_r} - 1} \cdot \mathbf{g}_{\psi^{-1}}(Q(\alpha_r^\vee) \cdot \langle y_\rho, \alpha_r \rangle)^{-1}.$$

Moreover, we can argue as in §4.2 that this condition is also sufficient. One has $\langle y, \alpha_r \rangle = 2r$ and thus $k_{y, \alpha_r} = 1$. The equality is thus simplified to

$$(20) \quad \overline{\chi}(\mathbf{s}_{-m\alpha_r^\vee}) = \mathbf{g}_{\psi^{-1}}(m)^{-1}.$$

Consider the exceptional character $\overline{\chi}_{\psi'} = \overline{\chi}_{\psi'}^0 \cdot \delta_B^{1/2n}$, which relies on the distinguished unitary character $\overline{\chi}_{\psi'}^0$ depending on a non-trivial character $\psi' : F \rightarrow \mathbf{C}^\times$ (see §2.3). Since $\overline{\chi}_{\psi'}^0(\mathbf{s}_{m\alpha_r^\vee}) = \gamma_{\psi'}(\varpi)^{mQ(\alpha_r^\vee)}$, by Lemma 4.3, the equality (20) becomes $\gamma_{\psi'}(\varpi) = (-1, \varpi)_n^{m^2} \cdot \gamma_{\psi'}(\varpi)^{-m}$, which can be further reduced to

$$\gamma_{\psi'}(\varpi) = (-1, \varpi)_n^{r+1} \cdot \gamma_{\psi'}(\varpi) = (-1, \varpi)_2^{r+1} \cdot \gamma_{\psi'}(\varpi).$$

In particular, if $\psi' = \psi_a$ with $a \in O_F^\times$, then the equality is equivalent to $(a(-1)^{r+1}, \varpi)_2 = -1$, i.e. $a \in (-1)^{r+1} \cdot (O_F^\times)^2$.

5.2.4. *For $m = 2r$.* We claim in this case $\mathcal{O}_{Q,n}^F = \mathcal{O}_{Q,n,\text{sc}}^F$. Clearly it suffices to show $\mathcal{O}_{Q,n}^F \supseteq \mathcal{O}_{Q,n,\text{sc}}^F$. Equivalently, if \mathcal{O}_y is not $Y_{Q,n}$ -free, we would like to show that it is not $Y_{Q,n}^{\text{sc}}$ -free. Write $\mathbf{i}_C^*(y_\rho) = (x_1^*, x_2^*, \dots, x_r^*)$. By assumption, $\mathbf{i}_C(y - \mathbf{w}[y]) = \mathbf{i}_C^*(y_\rho - \mathbf{w}(y_\rho)) \in Y_{Q,n}$ for some $\mathbf{w} \in W$. Entries of $\mathbf{i}_C(y - \mathbf{w}[y])$ can not be of the form $2x_i^* - 1$ since m is even; therefore they are of the form $0, x_i^* - x_j^*$ or $x_i^* + x_j^* - 1$ for $i \neq j$. In this case, it is easy to see that $\mathbf{i}_C(y - \mathbf{w}'[y]) \in Y_{Q,n}^{\text{sc}}$ for some $\mathbf{w}' \in W$, i.e., \mathcal{O}_y is not $Y_{Q,n}^{\text{sc}}$ -free.

Consequently, $\dim \text{Wh}_\psi(\Theta(\overline{\text{Sp}}_{2r}^{(4r)}, \overline{\chi})) = |\wp_{Q,n}(\mathcal{O}_{Q,n}^F)|$. On the other hand, consider \mathcal{O}_y with $\mathbf{i}_C^*(y_\rho) = (1, \dots, r-1, r)$. It is not hard to see $\wp_{Q,n}(\mathcal{O}_{Q,n}^F) = \{\wp_{Q,n}(\mathcal{O}_y)\}$. Therefore, we always have $\dim \text{Wh}_\psi(\Theta(\overline{\text{Sp}}_{2r}^{(4r)}, \overline{\chi})) = 1$ for any of the two exceptional characters of $\overline{\text{Sp}}_{2r}^{(4r)}$.

5.2.5. *For $m = 2r + 1$.* In this case, consider \mathcal{O}_y with $\mathbf{i}_C^*(y_\rho) = (1, 2, \dots, r-1, r)$. It can be checked that $\wp_{Q,n}(\mathcal{O}_{Q,n}^F) = \{\wp_{Q,n}(\mathcal{O}_y)\}$ with $\mathcal{O}_y \in \mathcal{O}_{Q,n}^F$, i.e. $|\wp_{Q,n}(\mathcal{O}_{Q,n}^F)| = 1$. On the other hand, $\wp_{Q,n}(\mathcal{O}_{Q,n,\text{sc}}^F) = \{\wp_{Q,n}(\mathcal{O}_y)\} \cup \{\wp_{Q,n}(\mathcal{O}_{z_i}) : 1 \leq i \leq r\}$ with z_i described as follows. Recall that we write $z_{i,\rho} := z_i - \rho$. For $1 \leq i \leq r-1$, z_i is such that $\mathbf{i}_C^*(z_{i,\rho}) = (0, 2, 3, \dots, \widehat{i+1}, \dots, r, r+1)$, which denotes the r -tuple obtained from the $r+1$ -tuple $(0, 2, 3, \dots, r-1, r, r+1)$ by removing the entry $i+1$. Meanwhile, z_r is such that $\mathbf{i}_C^*(z_{r,\rho}) = (2, 3, \dots, r-1, r, r+1)$.

Note $\mathcal{O}_{z_i} \in \mathcal{O}_{Q,n,\text{sc}}^F \setminus \mathcal{O}_{Q,n}^F$, since

$$\mathbf{i}_C(\mathbf{w}_{\alpha_r}[z_i] - z_i) = \mathbf{i}_C(\mathbf{w}_{\alpha_r}(z_{i,\rho}) - z_{i,\rho}) = -(0, 0, \dots, 0, m) = \mathbf{i}_C(-m\alpha_r^\vee) \in Y_{Q,n}.$$

The elements $\wp_{Q,n}(\mathcal{O}_y)$ and $\wp_{Q,n}(\mathcal{O}_{z_i})$'s are all distinct. It follows $|\wp_{Q,n}(\mathcal{O}_{Q,n,\text{sc}}^F)| = r + 1$. Therefore,

$$1 \leq \dim \text{Wh}_\psi(\Theta(\overline{\text{Sp}}_{2r}^{(4r+2)}, \bar{\chi})) \leq r + 1.$$

However, as there are only two exceptional characters $\bar{\chi}$, the dimension $\text{Wh}_\psi(\Theta(\overline{\text{Sp}}_{2r}^{(4r+2)}, \bar{\chi}))$ can take at most two values. In fact, we will determine completely the value and its dependence on $\bar{\chi}$.

Proposition 5.2. *Let $\bar{\chi}$ be an exceptional character of $\overline{\text{Sp}}_{2r}^{(4r+2)}$. Then*

$$\dim \text{Wh}_\psi(\Theta(\overline{\text{Sp}}_{2r}^{(4r+2)}, \bar{\chi})) = \begin{cases} 1 & \text{if } \bar{\chi}(\mathbf{s}_{-m\alpha_r^\vee}) = -q^{1/2} \cdot \gamma_\psi(\varpi), \\ r + 1 & \text{if } \bar{\chi}(\mathbf{s}_{-m\alpha_r^\vee}) = q^{1/2} \cdot \gamma_\psi(\varpi). \end{cases}$$

Proof. First, we show that if $\bar{\chi}$ is an exceptional character, then $\bar{\chi}(\mathbf{s}_{-m\alpha_r^\vee}) = \pm q^{1/2} \cdot \gamma_\psi(\varpi)$. Consider

$$\begin{aligned} & \bar{\chi}(\mathbf{s}_{-m\alpha_r^\vee})^2 \\ &= \bar{\chi}(\mathbf{s}_{-n\alpha_r^\vee}) \cdot \epsilon^{m^2 Q(\alpha_r^\vee)} \\ &= \bar{\chi}(\mathbf{s}_{n\alpha_r^\vee})^{-1} \cdot \epsilon \\ &= q \cdot (-1, \varpi)_2, \end{aligned}$$

which has square roots exactly $\pm q^{1/2} \cdot \gamma_\psi(\varpi)$. That is, an exceptional character $\bar{\chi}$ of $\overline{\text{Sp}}_{2r}^{(4r+2)}$ is uniquely determined by the sign.

Second, argue as in §4.2, we see that $\wp_{Q,n}(\mathcal{O}_{z_i}), 1 \leq i \leq r$ contributes to the right hand side of the Equality (10) if and only if (as in Equality (15))

$$(21) \quad \bar{\chi}(\mathbf{s}_{\mathbf{w}_{\alpha_r}[z_i] - z_i}) = \epsilon^{D(\mathbf{w}_{\alpha_r}[z_i] - z_i, z_i)} \cdot \mathbf{t}(\mathbf{w}_{\alpha_r}, z_i).$$

That is, $\dim \text{Wh}_\psi(\Theta(\overline{\text{Sp}}_{2r}^{(4r+2)}, \bar{\chi})) = 1 + |\{z_i : \text{the equality (21) holds for } z_i\}|$. Note, $\mathbf{w}_{\alpha_r}[z_i] - z_i = -m\alpha_r^\vee$ for all i . On the other hand, we claim that the right hand side of (21) are equal for all i . Simple computation gives $\langle z_{i,\rho}, \alpha_r \rangle = m$ and therefore

$$\begin{aligned} & \epsilon^{D(\mathbf{w}_{\alpha_r}[z_i] - z_i, z_i)} \cdot \mathbf{t}(\mathbf{w}_{\alpha_r}, z_i) \\ &= \epsilon^{D(\alpha_r^\vee, z_i)} \cdot q^{\left\lceil \frac{\langle z_{i,\rho}, \alpha_r \rangle + 1}{n\alpha_r} \right\rceil - 1} \cdot \epsilon^{\langle z_{i,\rho}, \alpha_r \rangle \cdot D(z_i, \alpha_r^\vee)} \cdot \mathbf{g}_{\psi^{-1}}(\langle z_{i,\rho}, \alpha_r \rangle \cdot Q(\alpha_r^\vee))^{-1} \\ &= \epsilon^{B_Q(z_i, \alpha_r^\vee)} \cdot q^{\left\lceil \frac{m+1}{n} \right\rceil - 1} \cdot \mathbf{g}_{\psi^{-1}}(m)^{-1} \\ &= \mathbf{g}_{\psi^{-1}}(m)^{-1} \text{ by the evenness of } B_Q. \end{aligned}$$

Thus, it follows that $\dim \text{Wh}_\psi(\Theta(\overline{\text{Sp}}_{2r}^{(4r+2)}, \bar{\chi})) = 1$ or $r + 1$. Moreover, it is equal to 1 if and only if $\bar{\chi}(\mathbf{s}_{-m\alpha_r^\vee}) \neq \mathbf{g}_{\psi^{-1}}(m)^{-1}$. By Lemma 4.3, $\mathbf{g}_{\psi^{-1}}(m)^{-1} = q^{1/2} \cdot \gamma_\psi(\varpi)$. Therefore, $\dim \text{Wh}_\psi(\Theta(\overline{\text{Sp}}_{2r}^{(4r+2)}, \bar{\chi})) = 1$ (resp. $r + 1$) if and only if $\bar{\chi}(\mathbf{s}_{-m\alpha_r^\vee})$ is equal to $-q^{1/2} \cdot \gamma_\psi(\varpi)$ (resp. $q^{1/2} \cdot \gamma_\psi(\varpi)$). The proof is completed. \square

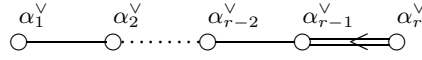
Theorem 5.3. Consider the Brylinski-Deligne covering group $\overline{Sp}_{2r}^{(n)}$ with $r \geq 2, n \geq 1$. Let $\overline{\chi}$ be an unramified exceptional character, then $\dim \text{Wh}_\psi(\Theta(\overline{Sp}_{2r}^{(n)}, \overline{\chi})) = 1$ if and only if the following holds:

- $n = 4r - 2$ and $\overline{\chi}$ is the unique exceptional character satisfying (20), or
- $n = 4r$ and $\overline{\chi}$ is any exceptional character of $\overline{Sp}_{2r}^{(4r)}$, or
- $n = 4r + 2$ and $\overline{\chi}$ is the unique exceptional character given in Proposition 5.2, or
- $n = 2r + 1$ and $\overline{\chi}$ is the only exceptional character of $\overline{Sp}_{2r}^{(2r+1)}$.

Moreover, consider the exceptional character $\overline{\chi}_{\psi_a} := \overline{\chi}_{\psi_a}^0 \cdot \delta_B^{1/2n}$ associated with ψ_a . Assume ψ_a has conductor O_F , i.e. $a \in O_F^\times$. Then, one has $\dim \text{Wh}_\psi(\Theta(\overline{Sp}_{2r}^{(4r-2)}, \overline{\chi}_{\psi_a})) = 1$ if and only if $a \in (-1)^{r+1} \cdot (O_F^\times)^2$; also, $\dim \text{Wh}_\psi(\Theta(\overline{Sp}_{2r}^{(4r+2)}, \overline{\chi}_{\psi_a})) = 1$ if and only if $a \in (-1)^r \cdot (O_F^\times)^2$.

6. THE $B_r, r \geq 2$ CASE

Consider the Dynkin diagram for the simple coroots for the group Spin_{2r+1} of type B_r :



Let $Y = \langle \alpha_1^\vee, \alpha_2^\vee, \dots, \alpha_{r-1}^\vee, \alpha_r^\vee \rangle$ be the cocharacter lattice of Spin_{2r+1} , where α_r^\vee is the long coroot. Let Q be the Weyl-invariant quadratic on Y such such $Q(\alpha_r^\vee) = 2$, i.e. $Q(\alpha_i^\vee) = 1$ for $1 \leq i \leq r-1$. Then the bilinear form B_Q is given by

$$B_Q(\alpha_i^\vee, \alpha_j^\vee) = \begin{cases} 4 & \text{if } i = j = r; \\ 2 & \text{if } 1 \leq i = j \leq r-1; \\ -1 & \text{if } 1 \leq i \leq r-2 \text{ and } j = i+1; \\ -2 & \text{if } i = r-1, j = r; \\ 0 & \text{if } \alpha_i^\vee, \alpha_j^\vee \text{ are not adjacent.} \end{cases}$$

The map $\mathbf{i}_B : \bigoplus_{i=1}^r \mathbf{Z}\alpha_i^\vee \rightarrow \bigoplus_{i=1}^r \mathbf{Z}e_i$ is given by

$$\mathbf{i}_B : (x_1, x_2, x_3, \dots, x_r) \mapsto (x_1, x_2 - x_1, x_3 - x_2, \dots, x_{r-1} - x_{r-2}, 2x_r - x_{r-1}).$$

In particular, any $(y_1, \dots, y_r) \in \bigoplus_{i=1}^r \mathbf{Z}e_i$ is equal to $\mathbf{i}_B(y)$ for some y if and only if $2 | (\sum_i y_i)$.

The Weyl group is $W = S_r \rtimes (\mathbf{Z}/2\mathbf{Z})^r$, where S_r is the permutation group on $\bigoplus_i \mathbf{Z}e_i$ and $(\mathbf{Z}/2\mathbf{Z})_i : e_i \mapsto \pm e_i$. In particular, $w_{\alpha_i}, 1 \leq i \leq r-1$, acts on $(y_1, y_2, \dots, y_r) \in \bigoplus_i \mathbf{Z}e_i$ by exchanging y_i and y_{i+1} . Also, w_{α_r} acts by (-1) on $\mathbf{Z}e_r$.

Simple computation gives:

$$Y_{Q,n} = \left\{ \begin{array}{l} (y_1, y_2, \dots, y_r) \in \bigoplus_{i=1}^r \mathbf{Z}e_i : \\ \bullet 2 | (\sum_{i=1}^r y_i), \\ \bullet y_1 \equiv \dots \equiv y_r \pmod{n} \\ \bullet n | 2y_i \text{ for all } i. \end{array} \right\}, \quad Y_{Q,n}^{\text{sc}} = \left\{ \begin{array}{l} (y_1, y_2, \dots, y_r) \in \bigoplus_{i=1}^r \mathbf{Z}e_i : \\ \bullet 2 | (\sum_{i=1}^r y_i), \\ \bullet n | y_i \text{ for all } i. \end{array} \right\}$$

Note in particular, if n is odd, $Y_{Q,n} = Y_{Q,n}^{\text{sc}}$.

We further note $2\rho = \sum_{i=1}^r 2(r-i+1)e_i$, and therefore $\rho = \sum_{i=1}^r (r-i+1)e_i$. If $y = (x_1, x_2, \dots, x_r) \in \bigoplus_i \mathbf{Z}\alpha_i^\vee$, then

$$\begin{aligned} \mathbf{i}_B(y_\rho) &= (x_1 - (r-1+1), x_2 - x_1 - (r-2+1), \dots, x_i - x_{i-1} - (r-i+1), \dots, \\ &\quad \dots, x_{r-1} - x_{r-2} - (r-(r-1)+1), 2x_r - x_{r-1} - (r-r+1)) \\ &:= (x_1^*, x_2^*, \dots, x_{r-1}^*, x_r^*). \end{aligned}$$

Any $(x_1^*, \dots, x_r^*) \in \bigoplus_i \mathbf{Z}e_i$ such that $2 | (\sum_i x_i^* + r(r+1)/2)$ is equal to $\mathbf{i}_B(y_\rho)$ for some y .

6.1. **For n odd.** In this case,

$$nY = Y_{Q,n}^{\text{sc}} = Y_{Q,n}.$$

Therefore, $\dim \text{Wh}_\psi(\Theta(\overline{\text{Spin}}_{2r+1}^{(n)}, \bar{\chi})) = |\wp_{Q,n}(\mathcal{O}_{Q,n,\text{sc}}^F)|$, where $\bar{\chi}$ is the only exceptional character of $\overline{\text{Spin}}_{2r+1}^{(n)}$. Note for n odd, the dual group for $\overline{\text{Spin}}_{2r+1}^{(n)}$ is PGSp_{2r} .

Proposition 6.1. *Let n be an odd number, one has*

$$\begin{cases} |\wp_{Q,n}(\mathcal{O}_{Q,n,\text{sc}}^F)| \geq 2 & \text{if } n \geq 2r+3; \\ |\wp_{Q,n}(\mathcal{O}_{Q,n,\text{sc}}^F)| = 0 & \text{if } n \leq 2r-1; \\ |\wp_{Q,n}(\mathcal{O}_{Q,n,\text{sc}}^F)| = 1 & \text{if } n = 2r+1. \end{cases}$$

Therefore, when n is odd, we have $\dim \text{Wh}_\psi(\Theta(\overline{\text{Spin}}_{2r+1}^{(n)}, \bar{\chi})) = 1$ if and only if $n = 2r+1$.

Proof. First, assume $n \geq 2r+3$. We write

$$\mathbf{i}_B(y_\rho) = (x_1^*, x_2^*, \dots, x_i^*, \dots, x_r^*) \text{ with } 2|(\sum_{i=1}^r x_i^* + r(r+1)/2).$$

For $r \geq 3$, let $y \in Y$ (resp. $y' \in Y$) be such that $\mathbf{i}_B(y_\rho) = (1, 2, 3, \dots, r-2, r-1, r)$ (resp. $\mathbf{i}_B(y'_\rho) = (1, 2, \dots, r-2, r, r+1)$). For $r = 2$, we take $(x_1^*, x_2^*) = (1, 2)$ or $(2, 3)$, and let y and y' be the associated element in Y respectively. In any case, the two orbits \mathcal{O}_y and $\mathcal{O}_{y'}$ are $Y_{Q,n}$ -free.

Moreover, $\wp_{Q,n}(\mathcal{O}_y) \neq \wp_{Q,n}(\mathcal{O}_{y'})$. Thus, for $n \geq 2r+3$, one has $|\wp_{Q,n}(\mathcal{O}_{Q,n,\text{sc}}^F)| \geq 2$.

Second, assume $n \leq 2r-1$, we want to show $\mathcal{O}_{Q,n,\text{sc}}^F = \emptyset$. If $\mathbf{i}_B(y_\rho) = (x_1^*, x_2^*, \dots, x_i^*, \dots, x_r^*)$ is such that $x_i^* \equiv x_j^* \pmod{n}$ for some $i \neq j$, then clearly $\mathcal{O}_y \notin \mathcal{O}_{Q,n,\text{sc}}^F$. Suppose $n \nmid (x_i^* - x_j^*)$ for all $i \neq j$, since $n \leq 2r-1$, it is not hard to see that there always exist i, j such that $n|(x_j^* + x_i^*)$. That is, $\mathcal{O}_y \notin \mathcal{O}_{Q,n,\text{sc}}^F$ for any \mathcal{O}_y .

Third, if $n = 2r+1$, consider the orbit \mathcal{O}_y with

$$\mathbf{i}_B(y_\rho) = (x_1^*, x_2^*, \dots, x_{r-1}^*, x_r^*) = (1, 2, 3, \dots, r-2, r-1, r).$$

(For $r = 2$, consider $\mathbf{i}_B(y_\rho) = (1, 2)$.) One has $\wp_{Q,n}(\mathcal{O}_{Q,n,\text{sc}}^F) = \{\wp_{Q,n}(\mathcal{O}_y)\}$, and therefore $|\wp_{Q,n}(\mathcal{O}_{Q,n,\text{sc}}^F)| = 1$ for $n = 2r+1$. \square

6.2. **For n even.** Write $n = 2m$. In this case,

$$Y = \left\{ (y_1, y_2, \dots, y_r) \in \bigoplus_i \mathbf{Z}e_i : 2|(\sum_{i=1}^r y_i) \right\}.$$

Moreover,

$$Y_{Q,n} = \left\{ \begin{array}{l} (y_1, y_2, \dots, y_r) \in \bigoplus_i \mathbf{Z}e_i : \\ \bullet 2|(\sum_{i=1}^r y_i), \\ \bullet y_i = k_i n + m \text{ for all } i \\ \text{or } y_i = k_i n \text{ for all } i. \end{array} \right\}, \quad Y_{Q,n}^{\text{sc}} = \left\{ \begin{array}{l} (y_1, y_2, \dots, y_r) \in \bigoplus_i \mathbf{Z}e_i : \\ \bullet n|y_i \text{ for all } i. \end{array} \right\}$$

We see easily that for $y_i = k_i n + m$, one has $(y_1, y_2, \dots, y_r) \in Y_{Q,n}$ if and only if $2|(rm)$. In fact, for n even, the dual group for $\overline{\text{Spin}}_{2r+1}^{(n)}$ is equal to SO_{2r+1} if m and r are both odd; otherwise, the dual group is Spin_{2r+1} , see [We2]. We discuss case by case according to the parities of r and m .

6.2.1. *For m odd and r odd.* In particular, one has $r \geq 3$. In this case, $Y_{Q,n} = Y_{Q,n}^{\text{sc}}$, and $\wp_{Q,n}(\mathcal{O}_{Q,n}^F) = \wp_{Q,n}(\mathcal{O}_{Q,n,\text{sc}}^F)$. Consider the following situations:

- If $n > 2(r+1)$ (i.e. $m > r+1$ and therefore $m \geq r+2$), consider y such that $\mathbf{i}_B(y_\rho) = (x_1^*, x_2^*, \dots, x_r^*)$ is equal to

$$(1, 2, \dots, r-2, r-1, r) \text{ or } (1, 2, \dots, r-2, r, r+1).$$

We can check the two orbits \mathcal{O}_y for these two choices of y are $Y_{Q,n}$ -free, and moreover their images with respect to the map $\wp_{Q,n}$ are distinct in $\wp_{Q,n}(\mathcal{O}_{Q,n}^F)$. Thus, $|\wp_{Q,n}(\mathcal{O}_{Q,n}^F)| \geq 2$ in this case.

- If $n < 2r$ (i.e. $m < r$ and therefore $m \leq r-2$), in this case, one can check $\wp_{Q,n}(\mathcal{O}_{Q,n,\text{sc}}^F) = \emptyset$.
- If $n = 2r$ (note $n \neq 2(r+1)$), i.e. $m = r$. In this case, one can also check $\wp_{Q,n}(\mathcal{O}_{Q,n,\text{sc}}^F) = \emptyset$.

Therefore, $\dim \text{Wh}_\psi(\Theta(\overline{\text{Spin}}_{2r+1}^{(n)}, \overline{\chi})) \neq 1$ for both r and m odd.

6.2.2. *For m odd and $r \geq 2$ even.* In this case, $Y_{Q,n} \neq Y_{Q,n}^{\text{sc}}$. One has the following situations:

- Assume $n > 2(r+1)$ (i.e. $m > r+1$ and thus $m \geq r+3$).

CASE I If $r \geq 3$, consider y and y' such that

$$\mathbf{i}_B(y_\rho) = (1, 2, \dots, r-2, r-1, r) \text{ and } \mathbf{i}_B(y'_\rho) = (1, 2, \dots, r-2, r, r+1).$$

We can check the orbits $\mathcal{O}_y, \mathcal{O}_{y'}$ are $Y_{Q,n}$ -free and $\wp_{Q,n}(\mathcal{O}_y) \neq \wp_{Q,n}(\mathcal{O}_{y'})$. Thus, $|\wp_{Q,n}(\mathcal{O}_{Q,n}^F)| \geq 2$.

CASE II If $r = 2$ and $m \geq r+5$, consider \mathcal{O}_y and $\mathcal{O}_{y'}$ with $\mathbf{i}_B(y_\rho) = (1, 2)$ and $\mathbf{i}_B(y'_\rho) = (2, 3)$. Then as in the preceding case, they are $Y_{Q,n}$ -free and $\wp_{Q,n}(\mathcal{O}_y) \neq \wp_{Q,n}(\mathcal{O}_{y'})$. Thus, $|\wp_{Q,n}(\mathcal{O}_{Q,n}^F)| \geq 2$.

CASE III If $r = 2$ and $m = 5$, consider \mathcal{O}_y with $\mathbf{i}_B(y_\rho) = (1, 2)$. It is easy to check $\wp_{Q,n}(\mathcal{O}_{Q,n}^F) = \{\wp_{Q,n}(\mathcal{O}_y)\}$. On the other hand, let z, z' be such that $\mathbf{i}_B(z_\rho) = (1, 4)$ and $\mathbf{i}_B(z'_\rho) = (2, 3)$. Then $\wp_{Q,n}(\mathcal{O}_{Q,n,\text{sc}}^F) = \{\wp_{Q,n}(\mathcal{O}_y), \wp_{Q,n}(\mathcal{O}_z), \wp_{Q,n}(\mathcal{O}_{z'})\}$, a set of size 3. Note, $\mathcal{O}_z, \mathcal{O}_{z'} \in \mathcal{O}_{Q,n,\text{sc}}^F \setminus \mathcal{O}_{Q,n}^F$. That is, $|\wp_{Q,n}(\mathcal{O}_{Q,n}^F)| = 1$ and $|\wp_{Q,n}(\mathcal{O}_{Q,n,\text{sc}}^F)| = 3$ in this case.

Let $\mathbf{w}, \mathbf{w}' \in W$ be such that $\mathbf{i}_B(\mathbf{w}[z] - z) = \mathbf{i}_B(\mathbf{w}'[z'] - z') = -(5, 5) \in Y_{Q,n}$. Write $y_{Q,n} = \mathbf{i}_B(\mathbf{w}[z] - z) \in Y_{Q,n}$. Then, as in §5.2.5, $\dim \text{Wh}_\psi(\Theta(\overline{\text{Spin}}_5^{(10)}, \overline{\chi})) = 1$ if and only if

$$(22) \quad \overline{\chi}(\mathbf{s}_{y_{Q,n}}) \neq \epsilon^{D(y_{Q,n}, z)} \cdot \mathbf{T}(\mathbf{w}, z) \text{ and } \overline{\chi}(\mathbf{s}_{y_{Q,n}}) \neq \epsilon^{D(y_{Q,n}, z')} \cdot \mathbf{T}(\mathbf{w}', z').$$

However, as in Proposition 5.2, it can be checked easily that $\epsilon^{D(y_{Q,n}, z)} \cdot \mathbf{T}(\mathbf{w}, z) = \epsilon^{D(y_{Q,n}, z')} \cdot \mathbf{T}(\mathbf{w}', z')$, and the condition (22) is equivalent to

$$(23) \quad \overline{\chi}(\mathbf{s}_{-5\alpha_r^\vee}) = -q^{1/2} \cdot \gamma_\psi(\varpi).$$

This agrees with the result from Proposition (5.2) for the $\overline{\text{C}}_2^{(10)}$ case.

- If $n < 2r$ (i.e. $m \leq r$ and therefore $m \leq r-1$), in this case, one can check $\wp_{Q,n}(\mathcal{O}_{Q,n,\text{sc}}^F) = \emptyset$.
- If $n = 2(r+1)$ (note $n \neq 2r$), i.e. $r = m-1$. In this case, one can check $\wp_{Q,n}^{\text{sc}}(\mathcal{O}_{Q,n,\text{sc}}^F) = \{\wp_{Q,n}^{\text{sc}}(\mathcal{O}_0)\}$ (and thus $\wp_{Q,n}(\mathcal{O}_{Q,n,\text{sc}}^F) = \{\wp_{Q,n}(\mathcal{O}_0)\}$) is a singleton with

$$\mathbf{i}_B(0_\rho) = (-r, -(r-1), \dots, -2, -1).$$

That is, \mathcal{O}_0 is $Y_{Q,n}^{\text{sc}}$ -free. However, it is not $Y_{Q,n}$ -free, since there exists $\mathbf{w} \in W$ such that $\mathbf{i}_B(\mathbf{w}(0_\rho)) = (1, 2, \dots, r-1, r)$. It follows

$$\mathbf{i}_B(\mathbf{w}(0_\rho) - 0_\rho) = (m, m, \dots, m, m) \in Y_{Q,n}.$$

Write $y_{Q,n} = \mathbf{w}(0_\rho) - 0_\rho = \mathbf{w}[0] - 0$. It follows from an analogous argument for Proposition 4.1 that $\dim \text{Wh}_\psi(\Theta(\overline{\text{Spin}}_{2r+1}^{(2r+2)}, \bar{\chi})) = 1$ if and only if $\bar{\chi}$ is the unique exceptional character satisfying

$$(24) \quad \bar{\chi}(\mathbf{s}_{y_{Q,n}}) = \mathbf{T}(\mathbf{w}, 0).$$

One can explicate the equality by computing the right hand side as in Lemma 4.2. We omit the details here.

6.2.3. *For m even and $r \geq 3$ odd.* In this case, $Y_{Q,n} \neq Y_{Q,n}^{\text{sc}}$. We have:

- If $n > 2(r+1)$ (i.e. $m > r+1$ and therefore $m \geq r+3$), consider y and y' such that

$$\mathbf{i}_B(y_\rho) = (1, 2, \dots, r-2, r-1, r) \text{ and } \mathbf{i}_B(y'_\rho) = (1, 2, \dots, r-2, r, r+1).$$

We can check the orbits $\mathcal{O}_y, \mathcal{O}_{y'}$ are $Y_{Q,n}$ -free and $\wp_{Q,n}(\mathcal{O}_y) \neq \wp_{Q,n}(\mathcal{O}_{y'})$. Thus, $|\wp_{Q,n}(\mathcal{O}_{Q,n}^F)| \geq 2$.

- If $n < 2r$ (i.e. $m < r$ and therefore $m \leq r-1$), in this case, one can check $\wp_{Q,n}(\mathcal{O}_{Q,n}^F) = \emptyset$.
- If $n = 2(r+1)$ (note $n \neq 2r$), i.e. $r = m-1$. In this case, $\wp_{Q,n}(\mathcal{O}_{Q,n}^F) = \{\wp_{Q,n}(\mathcal{O}_y)\}$ is a singleton with

$$\mathbf{i}_B(0_\rho) = (-r, -(r-1), \dots, -2, -1).$$

The situation is exactly as the third case of §6.2.2. That is, \mathcal{O}_0 is $Y_{Q,n}^{\text{sc}}$ -free but not $Y_{Q,n}$ -free. Consider $\mathbf{w} \in W$ such that $\mathbf{i}_B(\mathbf{w}(0_\rho)) = (1, 2, \dots, r-1, r)$ and

$$\mathbf{i}_B(\mathbf{w}(0_\rho) - 0_\rho) = (m, m, \dots, m, m) \in Y_{Q,n}.$$

Write $y_{Q,n} = \mathbf{w}(0_\rho) - 0_\rho = \mathbf{w}[0] - 0$. Then $\dim \text{Wh}_\psi(\Theta(\overline{\text{Spin}}_{2r+1}^{(2r+2)}, \bar{\chi})) = 1$ if and only if $\bar{\chi}$ is the unique exceptional character satisfying

$$(25) \quad \bar{\chi}(\mathbf{s}_{y_{Q,n}}) = \mathbf{T}(\mathbf{w}, 0).$$

6.2.4. *For m even and $r \geq 2$ even.* In this case, $Y_{Q,n} \neq Y_{Q,n}^{\text{sc}}$. One has the following situations:

- If $n > 2(r+1)$ (i.e. $m > r+1$ and therefore $m \geq r+2$).

CASE I, $r \geq 4$. Consider y and y' such that

$$\mathbf{i}_B(y_\rho) = (1, 2, \dots, r-2, r-1, r) \text{ and } \mathbf{i}_B(y'_\rho) = (1, 2, \dots, r-2, r, r+1).$$

We can check easily that the orbits \mathcal{O}_y and $\mathcal{O}_{y'}$ for these two choices are $Y_{Q,n}$ -free. Note, $|\wp_{Q,n}(\mathcal{O}_{Q,n}^F)| \geq 2$, since $\wp_{Q,n}(\mathcal{O}_y) \neq \wp_{Q,n}(\mathcal{O}_{y'})$.

CASE II, $r = 2$. Consider y and y' such that $\mathbf{i}_B(y_\rho) = (1, 2)$ and $\mathbf{i}_B(y'_\rho) = (2, 3)$. For $m \geq 4$, \mathcal{O}_y and $\mathcal{O}_{y'}$ are both $Y_{Q,n}$ -free. Moreover, we can check $\wp_{Q,n}(\mathcal{O}_{Q,n}^F) \subseteq \{\wp_{Q,n}(\mathcal{O}_y), \wp_{Q,n}(\mathcal{O}_{y'})\}$. Now if $m \geq 6$, then $\wp_{Q,n}(\mathcal{O}_y) \neq \wp_{Q,n}(\mathcal{O}_{y'})$. On the other hand, for $m = 4$, one has $\wp_{Q,n}(\mathcal{O}_y) = \wp_{Q,n}(\mathcal{O}_{y'})$ and therefore $\dim \text{Wh}_\psi(\Theta(\overline{\text{Spin}}_5^{(8)}, \bar{\chi})) = 1$ for any exceptional character $\bar{\chi}$ in this case.

To summarize for the case $m \geq r+2$:

$$\begin{cases} \dim \text{Wh}_\psi(\Theta(\overline{\text{Spin}}_{2r+1}^{(n)}, \bar{\chi})) = 1 & \text{if } m = 4, r = 2; \\ \dim \text{Wh}_\psi(\Theta(\overline{\text{Spin}}_{2r+1}^{(n)}, \bar{\chi})) \geq 2 & \text{if } r \geq 4 \text{ and } m \geq r+2, \text{ or } r = 2 \text{ and } m \geq 6. \end{cases}$$

- If $n < 2r$ (i.e. $m < r$ and therefore $m \leq r-2$), in this case, one can check easily $\wp_{Q,n}(\mathcal{O}_{Q,n}^F) = \emptyset$.
- If $n = 2r$ (note $n \neq 2(r+1)$), i.e. $r = m$, one also has $\wp_{Q,n}(\mathcal{O}_{Q,n}^F) = \emptyset$.

From the above discussion, we observe that for $r = 2$, the result agrees with that for covering groups of type C_2 , as expected. Therefore, we just summarize our result for covering $\overline{\text{Spin}}_{2r+1}^{(n)}$ with $r \geq 3$ as follows:

Theorem 6.2. Consider Brylinski-Deligne covering $\overline{\text{Spin}}_{2r+1}^{(n)}$ with $r \geq 3$. Let $\overline{\chi}$ be an exceptional character, then $\dim \text{Wh}_\psi(\Theta(\overline{\text{Spin}}_{2r+1}^{(n)}, \overline{\chi})) = 1$ if and only if one of the following holds:

- $n = 2(r+1)$, and $\overline{\chi}$ is the unique exceptional character satisfying (24) or (25);
- $n = 2r+1$, and $\overline{\chi}$ is the only exceptional character of $\overline{\text{Spin}}_{2r+1}^{(2r+1)}$.

7. THE G_2 CASE

Consider G_2 with Dynkin diagram for its simple coroots:

$$\begin{array}{c} \alpha_1^\vee \quad \alpha_2^\vee \\ \circ \quad \longleftrightarrow \quad \circ \end{array}$$

Let $Y = \langle \alpha_1^\vee, \alpha_2^\vee \rangle$ be the cocharacter lattice of G_2 , where α_1^\vee is the short coroot. Let Q be the Weyl-invariant quadratic on Y such such $Q(\alpha_1^\vee) = 1$ (thus $Q(\alpha_2^\vee) = 3$). Then the bilinear form B_Q is given by

$$B_Q(\alpha_i^\vee, \alpha_j^\vee) = \begin{cases} 2 & \text{if } i = j = 1; \\ -3 & \text{if } i = 1, j = 2; \\ 6 & \text{if } i = j = 2. \end{cases}$$

Simple computation gives:

$$Y_{Q,n} = Y_{Q,n}^{\text{sc}} = \mathbf{Z}(n_{\alpha_1} \alpha_1^\vee) \oplus \mathbf{Z}(n_{\alpha_2} \alpha_2^\vee),$$

where $n_{\alpha_2} = n/\gcd(n, 3)$ and $n_{\alpha_1} = n$.

The map $\mathbf{i}_G : \bigoplus_{i=1}^2 \mathbf{Z}\alpha_i^\vee \rightarrow \bigoplus_{i=1}^3 \mathbf{Z}e_i$ is given by

$$\mathbf{i}_G : (x_1, x_2) \mapsto (x_1 - 2x_2, x_2 - x_1, x_2).$$

Any $(y_i)_i \in \bigoplus_{i=1}^3 \mathbf{Z}e_i$ lies in the image of \mathbf{i}_G if and only if $y_1 + y_2 + y_3 = 0$.

The Weyl group $W = \langle \mathbf{w}_{\alpha_1}, \mathbf{w}_{\alpha_2} \rangle$ generated by \mathbf{w}_{α_1} and \mathbf{w}_{α_2} is the Dihedral group of order 12. In particular, $\mathbf{w}_{\alpha_1}(y_1, y_2, y_3) = (y_2, y_1, y_3) \in \bigoplus_{i=1}^3 \mathbf{Z}e_i$, and $\mathbf{w}_{\alpha_2}(y_1, y_2, y_3) = (-y_1, -y_3, -y_2)$.

By using \mathbf{i}_G , we could identify

$$Y_{Q,n} = Y_{Q,n}^{\text{sc}} = \left\{ \begin{array}{l} (y_1, y_2, y_3) \in \bigoplus_{i=1}^3 \mathbf{Z}e_i : \\ \bullet y_1 + y_2 + y_3 = 0, \\ \bullet y_1 \equiv y_2 \equiv y_3 \pmod{n}. \end{array} \right\}$$

We further note $\rho = 5\alpha_1^\vee + 3\alpha_2^\vee$ with $\mathbf{i}_G(\rho) = (-1, -2, 3) \in \bigoplus_{i=1}^3 \mathbf{Z}e_i$. It follows that for any $y = (x_1, x_2) \in \bigoplus_{i=1}^2 \mathbf{Z}\alpha_i^\vee$,

$$\mathbf{i}_G(y_\rho) = (x_1 - 2x_2 - 1, x_2 - x_1 - 2, x_2 + 3) \in \bigoplus_{i=1}^3 \mathbf{Z}e_i.$$

We may write $\mathbf{i}_G(y_\rho) = (x_1^*, x_2^*, x_3^*)$. In particular, $(x_1^*, x_2^*, x_3^*) \in \bigoplus_{i=1}^3 \mathbf{Z}e_i$ lies in the image of \mathbf{i}_G if and only if $x_1^* + x_2^* + x_3^* = 0$.

Note since $Y_{Q,n} = Y_{Q,n}^{\text{sc}}$, it follows $\dim \text{Wh}_\psi(\Theta(\overline{G}_2^{(n)}, \overline{\chi})) = |\wp_{Q,n}(\mathcal{O}_{Q,n}^F)|$, where $\overline{\chi}$ is the only exceptional character of $\overline{G}_2^{(n)}$ as $Z(\overline{G}_2^\vee) = 1$.

We proceed to determine the n such that $\dim \text{Wh}_\psi(\Theta(\overline{G}_2^{(n)}, \overline{\chi})) = 1$. We only give an outline of the argument, the details of which consists of basic combinatorial computations:

- For $n = 7, 8$ or $n \geq 10$, the orbit \mathcal{O}_y with $\mathbf{i}_G(y_\rho) = (-2, -1, 3)$ is $Y_{Q,n}$ -free.
- For $n = 8, 10, 11$ or $n \geq 13$, the orbit $\mathcal{O}_{y'}$ with $\mathbf{i}_G(y'_\rho) = (-3, -1, 4)$ is $Y_{Q,n}$ -free. Moreover, for $n = 8, 10, 11$ or $n \geq 13$, one has $\wp_{Q,n}(\mathcal{O}_y) \neq \wp_{Q,n}(\mathcal{O}_{y'})$ for $\mathbf{i}_G(y_\rho) = (-2, -1, 3)$ and $\mathbf{i}_G(y'_\rho) = (-3, -1, 4)$.
- If $\mathcal{O}_{Q,n}^F \neq \emptyset$, then necessarily $|Y/Y_{Q,n}^{\text{sc}}| \geq |W|$, i.e. $n \cdot n_{\alpha_2} \geq 12$. Thus $n \geq 4$.
- One can also check by hand $\mathcal{O}_{Q,n}^F = \emptyset$ for $n = 4, 5, 6, 9$.

- For $n = 7, 12$, $\wp_{Q,n}(\mathcal{O}_{Q,n}^F) = \{\wp_{Q,n}(\mathcal{O}_y)\}$ with $\mathbf{i}_G(y_\rho) = (-2, -1, 3)$. That is, we have $\dim \mathrm{Wh}_\psi(\Theta(\overline{G}_2^{(n)}, \overline{\chi})) = 1$ for $n = 7$ or 12 .

To summarize,

Theorem 7.1. *Consider the Brylinski-Deligne covering $\overline{G}_2^{(n)}$. Let $\overline{\chi}$ be the only exceptional character on $\overline{G}_2^{(n)}$, then $\dim \mathrm{Wh}_\psi(\Theta(\overline{G}_2^{(n)}, \overline{\chi})) = 1$ if and only if $n = 7$ or 12 .*

REFERENCES

- [BBF] B. Brubaker, D. Bump and S. Friedberg, *Weyl group multiple Dirichlet series, Eisenstein series and crystal bases*, Ann. of Math. **173** (2011), no. 2, 1081-1120.
- [BFH] D. Bump, S. Friedberg and J. Hoffstein, *Eisenstein series on the metaplectic group and nonvanishing theorems for automorphic L-functions and their derivatives*, Ann. of Math. (2) **131** (1990), no. 1, 53-127.
- [BJ] D. Ban and C. Jantzen, *The Langlands quotient theorem for finite central extensions of p-adic groups*, Glas. Mat. Ser. III **48** (68) (2013), no. 2, 313-334.
- [BD] J.-L. Brylinski and P. Deligne, *Central extensions of reductive groups by K_2* , I.H.E.S. Publ. Math., **94** (2001), 5-85.
- [BFG] D. Bump, S. Friedberg and D. Goldfeld (editors), *Multiple Dirichlet series, L-functions and automorphic forms*, Birkhäuser, 2012.
- [BH] D. Bump and J. Hoffstein, *On Shimura's correspondence*, Duke Math. J. **55** (3) (1987), 661-691.
- [BG] D. Bump and D. Ginzburg, *Symmetric square L-functions on $GL(r)$* , Ann. of Math. **136** (1) (1992), 137-205.
- [BGH] D. Bump, D. Ginzburg and J. Hoffstein, *The symmetric cube*, Inven. Math., **125** (1996), 413-449.
- [Bou] N. Bourbaki, *Lie groups and Lie algebras Chapters 4-6*, Springer, 2008.
- [CO] G. Chinta and O. Offen, *A metaplectic Casselman-Shalika formula for GL_r* , Amer. J. of Math. **135** (2013), 403-441.
- [CS] W. Casselman and F. Shahidi, *On irreducibility of standard modules for generic representations*, Ann. scient. Éc. Norm. Sup., **31** (1998), 561-589.
- [CS1] W. Casselman and J. Shalika, *The unramified principal series of p-adic groups. II. The Whittaker function*, Compo. Math. **41** (1980), No. 2, 207-231.
- [FG1] S. Friedberg and D. Ginzburg, *Criteria for the existence of cuspidal theta representations*, available at arxiv:1507.07413v1.
- [FG2] S. Friedberg and D. Ginzburg, *Theta functions on covers of symplectic groups*, available at arxiv:1601.04970v1.
- [FG3] S. Friedberg and D. Ginzburg, *Descent and theta functions for metaplectic groups*, to appear in J. European Math. Soc, also available at arXiv:1403.3930.
- [FGS] S. Friedberg, D. Goldberg and D. Szpruch, *Exceptional representations of higher covers of classical and similitude groups*, in preparation.
- [FL] M. Finkelberg and S. Lysenko, *Twisted geometric Satake equivalence*, J. Inst. Math. Jussieu **9**(4) (2010), 719-739.
- [Gao] F. Gao, *The Langlands-Shahidi L-functions for Brylinski-Deligne extensions*, preprint, to appear in Amer. J. Math.
- [GG] W. T. Gan and F. Gao, *The Langlands-Weissman program for Brylinski-Deligne extensions*, preprint, available at www.math.nus.edu.sg/~matgwt.
- [GHPS] S. Gelbart, R. Howe and I. Piatetski-Shapiro, *Uniqueness and existence of Whittaker models for the metaplectic group*, Israel J. of Math. **34** (1979), No. 1-2, 21-37.
- [GS] D. Goldberg and D. Szpruch, *Plancherel measures for coverings of p-adic $SL_2(F)$* , available at arxiv:1411.6134v2.
- [Gel] S. Gelbart, *Weil's representation and the spectrum of the metaplectic group*, Lec. Notes in Math. Vol. 530, Springer, 1976.
- [Gin] D. Ginzburg, *Certain conjectures relating unipotent orbits to automorphic representations*, Israel J. of Math. **151** (2006), 323-355.
- [Ka1] E. Kaplan, *The theta period of a cuspidal automorphic representation of $GL(n)$* , Int. Math. Res. Not. No. 8 (2015), 2168-2209.
- [Ka2] E. Kaplan, *Theta distinguished representations, inflation and the symmetric square L-function*, available at arxiv:1411.5051v3.
- [Ka3] E. Kaplan, *The characterization of theta-distinguished representations of GL_n* , available at arxiv:1502.06643v1.
- [KP] D. A. Kazhdan and S. J. Patterson, *Metaplectic forms*, I.H.E.S. Publ. Math. **59** (1984), 35-142.

- [Ma] H. Matsumoto, *Sur les sous-groupes arithmétiques des groupes semi-simples déployés*, Ann. Sci. École Norm. Sup. **4** (1969), 1-62.
- [Mc1] P. McNamara, *Principal series representations of metaplectic groups over local fields in Multiple Dirichlet series, L-functions and automorphic forms*, Birkhauser, 2012, 299-328.
- [Mc2] P. McNamara, *The metaplectic Casselman-Shalika formula*, Trans. of A.M.S. **368** No. 4 (2016), 2913-2937.
- [Mo] C. C. Moore, *Group extensions of p -adic and adelic linear groups*, I.H.E.S. Publ. Math., **35** (1968), 157-222.
- [MW] C. Moeglin and J. L. Waldspurger, *Médèles de Whittaker dégénérés pour des groupes p -adiques*, Mathe. Zeit. **196** (1987), 427-452.
- [Re] R. Reich, *Twisted geometric Satake equivalence via gerbes on the factorizable Grassmannian*, Rep. Theory **16** (2012), 345-449.
- [Rod] F. Rodier, *Whittaker models for admissible representations of reductive p -adic split groups in Harmonic analysis on homogeneous spaces*, Proc. Sympos. Pure Math., Vol. XXVI, A.M.S., Providence R.I., 1973, 425-430.
- [Sha] F. Shahidi, *Eisenstein series and automorphic L-functions*, A.M.S. Colloquium Publications **58**, 2010.
- [Shi] G. Shimura, *On modular forms of half integral weight*, Ann. of Math. **97** (1973), 440-481.
- [S] R. Steinberg, *Générateurs, relations et revêtements de groupes algébriques*, in *Colloq. Théorie des Groupes Algébriques* (Bruxelles, 1962), Louvain, 1962.
- [Suz1] T. Suzuki, *Distinguished representations of metaplectic groups*, Amer. J. of Math. **120** (4) (1998), 723-755.
- [Suz2] T. Suzuki, *On the Fourier coefficients of metaplectic forms*, Ryukyu Mathematical Journal, **25** (2012), 21-106.
- [Szp1] D. Szpruch, *The Langlands-Shahidi method for the metaplectic group and applications*, thesis (Tel Aviv University), available at arXiv:1004.3516v1.
- [Szp2] D. Szpruch, *Uniqueness of Whittaker model for the metaplectic group*, Pacific J. of Math. **232** (2007), No. 2, 453-469.
- [Szp3] D. Szpruch, *Some irreducibility theorems of parabolic induction on the metaplectic group via the Langlands-Shahidi method*, Israel J. of Math. **195** (2013), 897-971.
- [Tak] S. Takeda, *The twisted symmetric square L-function of $GL(r)$* , Duke Math. J. **163** (1) (2014), 175-266.
- [We1] M. Weissman, *Metaplectic tori over local fields*, Pacific J. of Math. **241**(1)(2009), 169-200.
- [We2] M. Weissman, *L-groups and parameters for covering groups*, to appear in *Astérisque*, available at <https://arxiv.org/pdf/1507.01042v2.pdf>.

DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, 150 N. UNIVERSITY STREET, WEST LAFAYETTE, IN 47907

E-mail address: gaofan.math@gmail.com